J. Fluid Mech. (1984), vol. 140, pp. 223–256
Printed in Great Britain

Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flowfield

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(Received 22 June 1983)

The flow of an idealized granular material consisting of uniform smooth, but inelastic, spherical particles is studied using statistical methods analogous to those used in the kinetic theory of gases. Two theories are developed: one for the Couette flow of particles having arbitrary coefficients of restitution (inelastic particles) and a second for the general flow of particles with coefficients of restitution near 1 (slightly inelastic particles). The study of inelastic particles in Couette flow follows the method of Savage & Jeffrey (1981) and uses an *ad hoc* distribution function to describe the collisions between particles. The results of this first analysis are compared with other theories of granular flow, with the Chapman–Enskog dense-gas theory, and with experiments. The theory agrees moderately well with experimental data and it is found that the asymptotic analysis of Jenkins & Savage (1983), which was developed for slightly inelastic particles, surprisingly gives results similar to the first theory even for highly inelastic particles. Therefore the 'nearly elastic' approximation is pursued as a second theory using an approach that is closer to the established methods of Chapman–Enskog gas theory. The new approach which determines the collisional distribution functions by a rational approximation scheme, is applicable to general flowfields, not just simple shear. It incorporates kinetic as well as collisional contributions to the constitutive equations for stress and energy flux and is thus appropriate for dilute as well as dense concentrations of solids. When the collisional contributions are dominant, it predicts stresses similar to the first analysis for the simple shear case.

1. Introduction

Several studies of the flow of granular materials have been based on the assumption that, under certain flow conditions, collisions between the particles provide the principal mechanism for the transport of properties such as momentum and energy (Ogawa, Umemura & Oshima 1980; Shen 1981; Savage & Jeffrey 1981; Jenkins & Savage 1983). This assumption naturally suggests an analogy with the Chapman–Enskog kinetic theory of dense gases (Chapman & Cowling 1970), and in fact certain of the statistical methods developed for gases have been used in the analysis of granular flow. Thus, to continue the analogy, we may speak of kinetic theories for granular flow. Savage & Jeffrey (1981) characterized their analysis by the fact that it applied to high shear rates; however, the term *kinetic theory* used here is more descriptive of the underlying assumptions.

Although the various theories have in common the assumption that collisions are the principal transport mechanism, each theory contains assumptions not shared by the others. Comparisons between them and with experiments are still possible, however, and are helpful in deciding upon ways in which the theories might be improved. Before making these comparisons, the theory of Savage & Jeffrey (1981) must be completed by allowing the granular particles to be inelastic. Here we have assumed that they possess a conventional coefficient of restitution e and calculated the energy dissipated as a result of collisions, so that the ratio of the velocity fluctuations to the mean flow – left undetermined in Savage & Jeffrey (1981) – can be found as a function of e.

In comparing the theories with each other, one sees that a merit of the Savage & Jeffrey treatment is its applicability for all values of e, while its weakness is its restriction to a mean simple shear flow. This is in contrast with the Jenkins & Savage (1983) treatment, which applies to any type of flow, but was developed for values of e near 1 (i.e. slightly inelastic particles); and even more in contrast with Chapman–Enskog theory, which applies only for $e = 1$.

Given that the aim is to produce constitutive relations for a general granular flow, consideration only of simple shear may seem to be too restrictive. However, we can use the analysis of the simple shear case, to make some assessment of the $1-e \ll 1$ approximation made by Jenkins & Savage (1983). The point of interest is that, to approximate their collision integrals, Jenkins & Savage followed the usual approach of kinetic gas theories and obtained asymptotic solutions corresponding to the case of small gradients of mean-flow quantities, quantities such as mean velocity and fluctuation kinetic energy. For the case of simple shear, this corresponds to small values of the Savage & Jeffrey R-parameter, where $R = (\sigma\, \mathrm{d}u/\mathrm{d}y)/\langle C^2\rangle^{\frac{1}{2}}$ was defined as the ratio of the characteristic mean shear velocity to the r.m.s. of the velocity fluctuations. In the context of inelastic particles this is *equivalent* to assuming $1-e \ll 1$. These kinds of asymptotic approaches, when used in kinetic gas theory, are found to give surprisingly accurate predictions for the transport coefficients. Typically, in these applications the temperatures are 'high' and the velocity gradients are 'small' in the sense that a parameter analogous to R is small compared with unity. On the other hand, for common granular flow systems, the validity of the assumption of $R \ll 1$ is not so clear. These systems are extremely dissipative. The velocity fluctuations do not maintain themselves, as would the temperature of a gas in an insulated vessel. In the absence of a mean rate of deformation field or some input of fluctuation kinetic energy to drive these velocity fluctuations, they will rapidly decay. Savage & Jeffrey (1981), on the basis of some comparisons of their predicted stresses with Couette-flow experiments, suggest that R is *not* small, but about one or two. The numerical modelling of Campbell & Brennen (1982) and the analysis of Shen (1981) corroborate this estimate of R. For such values of R, the pair-distribution function used by Jenkins & Savage (1983) can become negative and their small gradient asymptotic solutions may be subject to doubt. Thus there is the obvious question of how large R can be, before the asymptotic analyses of collision integrals, as done by Jenkins & Savage (1983), begin to depart significantly from more exact evaluations. This is one of the concerns of the present study.

The paper begins with a derivation of the integral forms for the constitutive equations, correcting some results given by Jenkins & Savage (1983). Since the analysis of general deformations is so complicated, attention will be restricted in the first part of the present paper to the Savage & Jeffrey case of a simple shear flow with no fluxes of fluctuation energy. The collision integrals for the stresses and rate of energy dissipation per unit volume are evaluated for arbitrary values of both R

and the coefficient of restitution e, both numerically and by using series expansions combined with other transformations. These results are compared with experimental measurements of Savage & Sayed (1980) and predictions of the granular-flow analyses of Jenkins & Savage (1983), Ogawa *et al.* (1980), Shen (1981) and the Chapman–Enskog dense-gas analysis (Chapman & Cowling 1970). The conclusion is that the range of e for which the results of the 'linearized' (small-R) theory are accurate is surprisingly large, and an extension that calculates the probability distributions rather than guesses at them is worth undertaking. Such an analysis is developed in the second part of the paper. Comparisons are made with the simpler *ad hoc* analysis of Jenkins & Savage (1983) which assumes simple expressions for the singlet- and pair-distribution functions.

2. Integral forms for the constitutive equations for stress, flux of fluctuation energy and dissipation

The procedure we shall use for the development of the so-called *hydrodynamic* or *fluid-mechanical* equations and the integral forms for stresses, energy fluxes and the collisional rate of energy dissipation per unit volume is similar to the approach employed in the kinetic theory of dense fluids (Chapman & Cowling 1970; Ferziger & Kaper 1972). As noted previously, Savage & Jeffrey (1981) and Jenkins & Savage (1983) have considered the flows of granular materials composed of smooth, hard, spherical particles in this context. Some manipulations of the collisional integrals in the paper of Jenkins & Savage (1983) are inappropriate and some higher-order terms were neglected. Although these oversights did not affect the end results for the constitutive equations of Jenkins & Savage (1983), they would cause errors in the more detailed analysis given in the latter part of the present paper.

In the present section we provide a more careful derivation of these collisional integrals for the case of smooth, hard, inelastic particles of uniform diameter. Those steps that closely follow the papers of Savage & Jeffrey (1981) and Jenkins & Savage (1983) are only outlined to the extent necessary to define terminology.

We consider a fixed volume element $\mathrm{d}\boldsymbol{r}$ centred at $\boldsymbol{r}$ that contains $n(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{r}$ particles, where n is the number density of particles. The ensemble average of the single-particle quantity ψ is defined as

$$\langle\psi\rangle = \frac{1}{n}\int \psi f^{(1)}(\boldsymbol{r}, \boldsymbol{c}; t)\,\mathrm{d}\boldsymbol{c}, \tag{2.1}$$

where $f^{(1)}(\boldsymbol{r}, \boldsymbol{c}; t)$ is the usual single-particle velocity distribution function defined such that $f^{(1)}\,\delta\boldsymbol{c}$ is the differential number of particles per unit volume with velocities within the range $\boldsymbol{c}$ and $\boldsymbol{c}+\delta\boldsymbol{c}$. Similarly, a complete pair-distribution function $f^{(2)}(\boldsymbol{r}_1, \boldsymbol{c}_1; \boldsymbol{r}_2, \boldsymbol{c}_2; t)$ may be defined such that $f^{(2)}(\boldsymbol{r}_1, \boldsymbol{c}_1; \boldsymbol{r}_2, \boldsymbol{c}_2; t)\,\delta\boldsymbol{r}_1\,\delta\boldsymbol{r}_2\,\delta\boldsymbol{c}_1\,\delta\boldsymbol{c}_2$ is the probability of finding a pair of particles in the volume elements $\delta\boldsymbol{r}_1, \delta\boldsymbol{r}_2$ centred on the points $\boldsymbol{r}_1, \boldsymbol{r}_2$ and having velocities within the ranges $\boldsymbol{c}_1$ and $\boldsymbol{c}_1+\delta\boldsymbol{c}_1$, and $\boldsymbol{c}_2$ and $\boldsymbol{c}_2+\delta\boldsymbol{c}_2$. The rate of change of $\langle n\psi\rangle$ may be expressed as (Reif 1965)

$$\frac{\partial}{\partial t}\langle n\psi\rangle = n\langle \mathrm{D}\psi\rangle - \boldsymbol{\nabla}\cdot\langle n\boldsymbol{c}\psi\rangle + \phi_{\mathrm{c}}, \tag{2.2}$$

where

$$\mathrm{D}\psi = \frac{\mathrm{d}\boldsymbol{c}}{\mathrm{d}t}\cdot\frac{\partial\psi}{\partial\boldsymbol{c}} = \frac{\boldsymbol{F}}{m}\cdot\frac{\partial\psi}{\partial\boldsymbol{c}}, \tag{2.3}$$

$\boldsymbol{F}$ is the external force acting on a particle of mass m, and ϕ_{c} is the collisional rate of increase of the mean of ψ per unit volume.

To obtain more explicit expressions for the collisional term ϕ_c, we consider binary collisions between hard, smooth but inelastic particles of uniform diameter σ. At the instant of a collision between particles labelled 1 and 2 we take the centre O_2 of the second particle to be located at position $\boldsymbol{r}$ and the centre O_1 of particle 1 to be at $\boldsymbol{r}-\sigma\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector along the line of centres from O_1 to O_2. In time δt prior to the collision, particle 1 moved through a distance $\boldsymbol{c}_{12}\,\delta t$ relative to particle 2, where $\boldsymbol{c}_{12}=\boldsymbol{c}_1-\boldsymbol{c}_2$. For a collison to occur within δt, then, O_1 must lie within the volume $\sigma^2\delta\boldsymbol{k}(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\delta t$.

The probable number of collisions such that O_2 lies within the volume $\delta\boldsymbol{r}$ and in which $\boldsymbol{c}_1, \boldsymbol{c}_2$ and $\boldsymbol{k}$ lie within the ranges $\delta\boldsymbol{c}_1$, $\delta\boldsymbol{c}_2$ and $\delta\boldsymbol{k}$ is

$$\sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\,\boldsymbol{r},\boldsymbol{c}_2;t)\,\delta\boldsymbol{k}\,\delta\boldsymbol{c}_1\,\delta\boldsymbol{c}_2\,\delta\boldsymbol{r}\,\delta t.$$

During a collision, particle 2 gains a quantity $\psi_2'-\psi_2$ of ψ, where primed and unprimed quantities refer to values after and before the collision. Considering only particles that are about to collide (i.e. taking $\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0$), we find that the collisional rate of increase for the mean of ψ per unit volume

$$\phi_c=\sigma^2\int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0}(\psi_2'-\psi_2)\,(\boldsymbol{c}_{12}\cdot\boldsymbol{k})f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\boldsymbol{r},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{2.4}$$

This may be put in a more convenient and physically significant form. Interchanging the roles of the colliding particles by interchanging the subscripts 1 and 2 and replacing $\boldsymbol{k}$ by $-\boldsymbol{k}$, (2.4) may be written as

$$\phi_c=\sigma^2\int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0}(\psi_1'-\psi_1)\,(\boldsymbol{c}_{12}\cdot\boldsymbol{k})f^{(2)}(\boldsymbol{r},\boldsymbol{c}_1;\boldsymbol{r}+\sigma\boldsymbol{k},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{2.5}$$

After expanding the pair-distribution function $f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\,\boldsymbol{r},\boldsymbol{c}_2;t)$ in a Taylor series and rearranging, we may write

$$\begin{aligned}
&f^{(2)}(\boldsymbol{r},\boldsymbol{c}_1;\boldsymbol{r}+\sigma\boldsymbol{k},\boldsymbol{c}_2;t)\\
&\quad=f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\boldsymbol{r},\boldsymbol{c}_2;t)\\
&\qquad+[\sigma\boldsymbol{k}\cdot\nabla-\frac{1}{2!}(\sigma\boldsymbol{k}\cdot\nabla)^2+\frac{1}{3!}(\sigma\boldsymbol{k}\cdot\nabla)^3+\ldots]\,f^{(2)}(\boldsymbol{r},\boldsymbol{c}_1;\boldsymbol{r}+\sigma\boldsymbol{k},\boldsymbol{c}_2;t).
\end{aligned} \tag{2.6}$$

Substituting (2.6) in (2.5) and adding this to (2.4), we obtain

$$\phi_c=-\nabla\cdot\boldsymbol{\theta}+\chi, \tag{2.7}$$

where the collisional transfer contribution

$$\begin{aligned}
\boldsymbol{\theta}=-\frac{\sigma^3}{2}\int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0}(\psi_1'-\psi_1)\,(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\boldsymbol{k}\left[1-\tfrac{1}{2}\sigma\boldsymbol{k}\cdot\nabla+\frac{1}{3!}(\sigma\boldsymbol{k}\cdot\nabla)^2+\ldots\right]\\
\times f^{(2)}(\boldsymbol{r},\boldsymbol{c}_1;\boldsymbol{r}+\sigma\boldsymbol{k},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2,
\end{aligned} \tag{2.8}$$

and the 'source-like' contribution

$$\chi=\frac{\sigma^2}{2}\int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0}(\psi_2'+\psi_1'-\psi_2-\psi_1)\,(\boldsymbol{c}_{12}\cdot\boldsymbol{k})f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\boldsymbol{r},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{2.9}$$

Note that, when ψ is a summational invariant, χ is zero. When ψ is taken to be the translational kinetic energy $\frac{1}{2}mc^2$, then $\chi=-\gamma$, where γ is the rate of energy dissipation per unit volume due to inelastic collisions of the smooth particles. Also note that to first order

$$\boldsymbol{\theta} \approx -\tfrac{1}{2}\sigma^3 \int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0} (\psi_1' - \psi_1)(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\boldsymbol{k} f^{(2)}(\boldsymbol{r}-\tfrac{1}{2}\sigma\boldsymbol{k}, \boldsymbol{c}_1; \boldsymbol{r}+\tfrac{1}{2}\sigma\boldsymbol{k}, \boldsymbol{c}_2; t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2, \tag{2.10}$$

which is the same expression as given by Chapman & Cowling (1970, §16.4) for the collisional transfer of 'molecular' properties. Equations (2.2), (2.9) and (2.10) have been derived previously by Condiff, Lu & Dahler (1965) using slightly different arguments. The expression for $\boldsymbol{\theta}$ given by (2.8) differs from the corresponding expression of Jenkins & Savage (1983); the difference is due to an incorrect spatial shifting of $f^{(2)}$ and the neglect of some higher-order terms in Jenkins & Savage (1983). Because of the simple form used by Jenkins & Savage for $f^{(2)}$, their final integrated expressions for the constitutive equations are no different from those that are obtained using (2.8) and (2.9). However, for more general forms for $f^{(2)}$ there would be differences. It is worth mentioning that Brush (1972, pp. 72–76), in his book on the history of kinetic theory, has described the controversy concerning the appropriate form for Boltzmann-like equations and the collisional-transfer contributions for dense fluids in which the forms used by Enskog were first rejected and then reinstated. The present correction is more pedestrian, but nevertheless analogous to the issues described by Brush.

Considering the particles to be smooth but inelastic with a coefficient of restitution e (Jenkins & Savage 1983), one finds that the component of relative velocity $\boldsymbol{c}_{12}$ in the direction of $\boldsymbol{k}$ is changed during a collision such that

$$\boldsymbol{k}\cdot\boldsymbol{c}_{12}' = -e(\boldsymbol{k}\cdot\boldsymbol{c}_{12}). \tag{2.11}$$

The translational kinetic-energy change during a collision is

$$\begin{aligned}\Delta E &= \tfrac{1}{2}m[(c_1'^2 + c_2'^2) - (c_1^2 + c_2^2)] \\ &= -\tfrac{1}{4}m(1-e^2)(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^2.\end{aligned} \tag{2.12}$$

By taking ψ to be m, $m\boldsymbol{c}$ and $\tfrac{1}{2}mc^2$ in (2.2) and (2.7)–(2.9) we obtain the usual hydrodynamic equations

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{2.13}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \rho\boldsymbol{b} - \boldsymbol{\nabla}\cdot\boldsymbol{p}, \tag{2.14}$$

$$\tfrac{3}{2}\rho\frac{\mathrm{d}T}{\mathrm{d}t} = -\boldsymbol{p}:\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{q} - \gamma. \tag{2.15}$$

In these equations $\rho = mn = \nu\rho_{\mathrm{p}}$ is the bulk mass density, ν is the bulk solids fraction, ρ_{p} is the mass density of an individual particle, $\boldsymbol{u} = \langle\boldsymbol{c}\rangle$ is the bulk velocity, $\boldsymbol{p}$ is the pressure tensor, $\gamma = -\chi(\tfrac{1}{2}mc^2)$ is the collisional rate of dissipation per unit volume and $\boldsymbol{b}$ is the body force per unit mass. Finally, $\tfrac{3}{2}T = \tfrac{1}{2}\langle C^2\rangle$ is the specific kinetic energy of the velocity fluctuations (or for short the translational fluctuation energy) where $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{u}$, and $\boldsymbol{q}$ is the flux of fluctuation energy. The quantity T has become known as the granular temperature.

The total pressure tensor $\boldsymbol{p}$ is the sum of a kinetic part

$$\boldsymbol{p}_{\mathrm{k}} = \rho \langle \boldsymbol{C}\boldsymbol{C} \rangle \tag{2.16}$$

and a collisional part

$$\boldsymbol{p}_{\mathrm{c}} = \boldsymbol{\theta}(m\boldsymbol{C}). \tag{2.17}$$

Similarly, the flux of fluctuation energy $\boldsymbol{q}$ is the sum of a kinetic part

$$\boldsymbol{q}_{\mathrm{k}} = \tfrac{1}{2}\rho \langle C^2 \boldsymbol{C} \rangle \tag{2.18}$$

and a collisional part

$$\boldsymbol{q}_{\mathrm{c}} = \boldsymbol{\theta}(\tfrac{1}{2}mC^2). \tag{2.19}$$

3. Couette flow of inelastic particles

In this section we specialize to the case of a mean simple shear flow having no gradients of fluctuation kinetic energy. This is the case considered by Savage & Jeffrey (1981), and we use the pair-distribution function they proposed. Although their distribution function applies only to shear flow, it has the advantage that it does not require the quantities R or e to fall within any particular range. This same flow has been investigated theoretically by Ogawa *et al.* (1980) and Shen (1981); also, it probably is a moderately good idealization of the shear-cell experiments of Savage & Sayed (1980). Comparisons of the various granular-flow theories with experiments and the Chapman–Enskog dense-gas analysis (Chapman & Cowling 1970) will give some ideas of their merits and shortcomings.

3.1. *Analysis*

We consider the case of a simple shear flow (constant shear rate) $\boldsymbol{u} = u(y)\,\boldsymbol{e}_x$ of uniform bulk density ρ and uniform granular temperature T. In this flow $\boldsymbol{q} = 0$, and the translational fluctuation energy equation (2.15) reduces to a simple balance between the shear work and the rate of dissipation

$$p_{xy}\frac{\mathrm{d}u}{\mathrm{d}y} + \gamma = 0. \tag{3.1}$$

In order to evaluate the stress tensor $\boldsymbol{p}$ and the rate of dissipation $\gamma = -\chi(\tfrac{1}{2}mc^2)$ we shall make use of the complete pair-distribution function assumed by Savage & Jeffrey (1981). We write this down using their parameter R, or equivalently the temperature $T = \tfrac{1}{3}\langle v^2 \rangle$, and only later derive the expression for the parameter R in terms of e by appealing to (3.1). To be consistent with most of the other granular-flow theories we now use the symbol $\boldsymbol{v}$ to designate the peculiar velocity, which in the previous section was specified as $\boldsymbol{C}$. The coordinate system is that defined by Savage & Jeffrey (1981, figure 4) with the modification that ϑ here will be $\tfrac{1}{2}\pi - \vartheta$ as defined there. Thus if $\boldsymbol{k}$ is the unit vector along the line of centres, we employ

$$f^{(2)}(\boldsymbol{r}_1, \boldsymbol{c}_1; \boldsymbol{r}_2, \boldsymbol{c}_2; t) = \frac{n^2}{(2\pi T)^3} g_0(\nu)\,\mathrm{erfc}\,(\Phi)\exp\left(-\frac{(\boldsymbol{c}_1-\boldsymbol{u}_1)^2 + (\boldsymbol{c}_2-\boldsymbol{u}_2)^2}{2T}\right), \tag{3.2}$$

where

$$\Phi = \frac{\sigma \boldsymbol{k}\boldsymbol{k} : \nabla \boldsymbol{u}}{2T^{\frac{1}{2}}} = \frac{\surd 3}{2} R \sin\vartheta \cos\vartheta \cos\varphi,$$

and $g_0(\nu)$ is the Carnahan–Starling (1969) expression for the radial distribution function at contact:

$$g_0(\nu) = \frac{1}{1-\nu} + \frac{3\nu}{2(1-\nu)^2} + \frac{\nu^2}{2(1-\nu)^3}. \tag{3.3}$$

Recent computer experiments by Campbell (1982) show a similar form for the collisional distribution function at low concentration; however, he finds a distribution function that contained spikes at higher concentrations.

For a collision between two identical, smooth, inelastic, spherical particles of diameter σ and mass m, the conservation of linear momentum requires that the velocities of the particles before collision $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are related to those after collision by

$$m\boldsymbol{c}_1' = m\boldsymbol{c}_1 - \boldsymbol{J}, \tag{3.4a}$$

$$m\boldsymbol{c}_2' = m\boldsymbol{c}_2 + \boldsymbol{J}, \tag{3.4b}$$

where $\boldsymbol{J}$ is the impulse of the force between the particles. The coefficient of restitution e is introduced through (2.11), allowing us to eliminate $\boldsymbol{J}$ from (3.4) to obtain

$$\boldsymbol{c}_1' = \boldsymbol{c}_1 - \tfrac{1}{2}(1+e)(\boldsymbol{k}\cdot\boldsymbol{c}_{12})\boldsymbol{k}, \tag{3.5a}$$

$$\boldsymbol{c}_2' = \boldsymbol{c}_2 + \tfrac{1}{2}(1+e)(\boldsymbol{k}\cdot\boldsymbol{c}_{12})\boldsymbol{k}. \tag{3.5b}$$

Thus from the general expressions (2.10) and (2.17) with $\psi = m\boldsymbol{C}$, we see that the expression for the stress tensor is modified from Savage & Jeffrey (1981) only by the factor $\frac{1}{2}(1+e)$:

$$\boldsymbol{p} = \tfrac{1}{4}(1+e)\,m\sigma^3 \int_{\boldsymbol{k}\cdot\boldsymbol{c}_{12}>0} (\boldsymbol{k}\cdot\boldsymbol{c}_{12})^2\,\boldsymbol{kk}f^{(2)}(\boldsymbol{r}-\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_1;\boldsymbol{r}+\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{3.6}$$

Similarly, using (2.9), we obtain

$$\gamma = \tfrac{1}{8}(1-e^2)\,m\sigma^2 \int_{\boldsymbol{k}\cdot\boldsymbol{c}_{12}>0} (\boldsymbol{k}\cdot\boldsymbol{c}_{12})^3 f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\boldsymbol{r},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{3.7}$$

We non-dimensionalize the stress and energy dissipation by

$$\boldsymbol{p}^* = 2\boldsymbol{p}/(1+e)\,\rho\nu\, g_0(\nu)\,\sigma^2\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2, \tag{3.8}$$

$$\gamma^* = \gamma/(1-e^2)\,\rho\nu\, g_0(\nu)\,\sigma^2\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3, \tag{3.9}$$

where $\rho = nm$ and is the bulk solid density. We now retrace the steps taken in Savage & Jeffrey (1981) to simplify the integrals, with the exception that ϑ will here equal $\frac{1}{2}\pi - \vartheta$ as they define it. We obtain

$$\boldsymbol{p}^* = \frac{4\pi^{-\frac{3}{2}}}{R^2}\int_0^\infty\int_0^{2\pi}\int_0^\pi \zeta^2\boldsymbol{kk}\,\mathrm{erfc}\,(\Phi)\exp\left(-(\zeta+\Phi)^2\right)\cos\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\zeta, \tag{3.10}$$

$$\gamma^* = \frac{2\pi^{-\frac{3}{2}}}{3^{\frac{1}{2}}R^3}\int_0^\infty\int_0^{2\pi}\int_0^\pi \zeta^3\,\mathrm{erfc}\,(\Phi)\exp\left(-(\zeta+\Phi)^2\right)\cos\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\zeta, \tag{3.11}$$

where $\zeta = \frac{1}{2}\boldsymbol{k}\cdot\boldsymbol{c}_{12}\,T^{-\frac{1}{2}}$, and Φ was introduced in (3.2).

The numerical evaluation of (3.10) as a function of R is unchanged from Savage & Jeffrey (1981), because all new factors have been incorporated in the non-dimensionalization, and the evaluation of (3.11) for γ^* is similar. The simple and smooth shapes taken by all the non-dimensional functions allows us to express all the results using simple mimic functions which predict the numerical results to within 3 significant figures for all values of R. They are chosen by evaluating the integrals in (3.10) and (3.11) asymptotically for small and large values of R and then joining these extremes through the use of transformed R-variables. We define 4 subsidiary variables

$$\rho_n = \frac{R^2}{R^2+d_n}, \tag{3.12}$$

where $n = 1, 2, 3, 4$ and the d_n will be chosen separately for each function. A functional form that reproduces the series expansion for p^*_{xx} correct to $O(R^4)$ is

$$p^*_{xx} = P^*_{yy} = \frac{4}{3R^2}+\frac{6(4+\pi)}{35\pi}-\frac{6(4-\pi)}{35\pi}\rho_1, \tag{3.13}$$

and if $d_1^2 = 858(4-\pi)$ the correct value of $\frac{12}{35}$ is obtained as $R \to \infty$. Similarly,

$$p^*_{zz} = \frac{4}{3R^2}+\frac{2(4+\pi)}{35\pi}-\frac{2(4-\pi)}{35\pi}\rho_2, \tag{3.14}$$

where $d_2^2 = \frac{2574}{7}(4-\pi)$, to obtain $\frac{4}{35}$ as $R \to \infty$. Next a functional form that is correct to $O(R^3)$ for p^*_{xy} is

$$p^*_{xy} = -\frac{(d_3\rho_3)^{-\frac{1}{2}}}{5(3\pi)^{\frac{1}{2}}}[12+(d_3-6)\rho_3-(\tfrac{285}{8008}d_3^2-\tfrac{1}{2}d_3+\tfrac{3}{2})\rho_3^2], \tag{3.15}$$

and to obtain $-32/35\pi$ as $R \to \infty$ we need $d_3 = 20.6474$. Similarly, a functional form that is correct to $O(R^3)$ for γ^* is

$$\gamma^* = \frac{(d_4\rho_4)^{-\frac{3}{2}}}{(3\pi)^{\frac{1}{2}}}[4+(\tfrac{6}{5}d_4-6)\rho_4+(\tfrac{9}{280}d_4^2-\tfrac{3}{5}d_4+\tfrac{3}{2})\rho_4^2+(-\tfrac{9}{8008}d_4^3+\tfrac{9}{560}d_4^2-\tfrac{3}{20}d_4+\tfrac{1}{4})\rho_4^3], \tag{3.16}$$

and to obtain $8/35\pi$ as $R \to \infty$ we take $d_4 = 0.888348$.

3.2. *The relation between R and e*

From (3.1), (3.8) and (3.9) we obtain the equation

$$e = 1-\frac{1}{2}\frac{p^*_{xy}}{\gamma^*}, \tag{3.17}$$

from which we obtain e as a function of R. In physical terms, of course, we think more of R as a function of e. The relation is plotted in figure 1 together with data from §3.3. It can be seen that, although Savage & Jeffrey (1981) anticipated values of R and 0 and infinity, it is only for $R > 2.73$ that physically possible values of e are obtained. This result points out the simplicity of the dissipation mechanism assumed in the present model. It is obvious that in a granular material there are many additional dissipation mechanisms that have been omitted here. Thus one would expect that the results in figure 1 represent a low estimate of R for any given value of e. In particular, the experiments of Bagnold (1954), which probably achieved much higher values of R, had a suspending medium of the same density as the particles, which would have changed the dissipation mechanisms greatly from the one considered here.

3.3. *Comparisons with other theories and experiments*

In this section we shall make comparisons of the results just obtained with predictions given by the recent granular-flow theories put forth by Ogawa *et al.* (1980), Shen (1981) and Jenkins & Savage (1983). All of these are microstructural theories based upon collisional transfer of momentum and energy, but they involve different

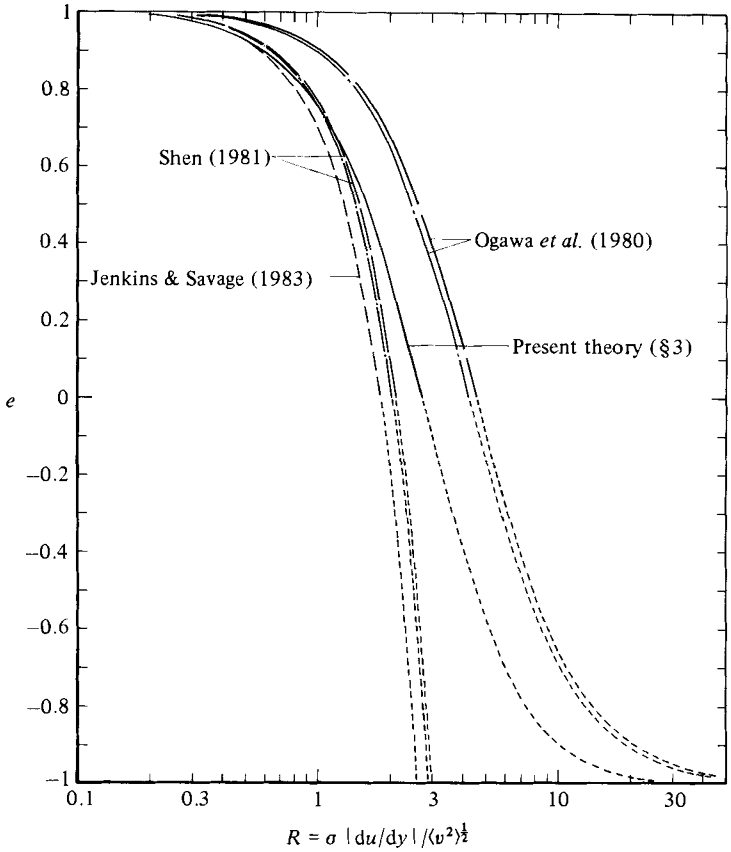

FIGURE 1. Coefficient of restitution e that is required to obtain a specified value of R predicted by various theories for the case of simple shear. The curves for the present §3 theory and for Jenkins & Savage (1983) are the same for all values of ν. The two curves for Ogawa *et al.* (1980) and for Shen (1981) are given for $\nu = 0.5$; they differ in the assumed value for ν^*, the 'maximum packing'. — —, $\nu^* = 0.64$; —-—, $\nu^* = 0.74$.

assumptions about the material properties and use different methods for the statistical averaging and evaluation of the collisional integrals. In addition, they each require further manipulation to recast them in a form suitable for comparison with the present work. The predictions of stresses for the case of simple shear are also compared with experimental results for the annular shear-cell tests of Savage & Sayed (1980) and with the Chapman–Enskog dense-gas kinetic theory (Chapman & Cowling 1970).

3.3.1. *Ogawa, Umemura & Oshima* (1980)

Ogawa *et al.* (1980) determined the stress tensor and the rate of energy dissipation for the flow of adhesive, rough, inelastic spherical granular particles by employing a simple statistical model of particle collisional interactions. In their model the friction of the particles was considered but particle rotations and rotary inertia were ignored. Furthermore, the fluctuations were assumed to be random and isotropic, and, although the existence of fluxes of fluctuations were recognized, such fluxes were later neglected in the determinations of the constitutive equations for the stresses and rate of energy dissipation.

Each particle was considered to be inside an imaginary collision sphere of radius b which represented the 'wall' being set up by the neighbouring particles. Multiplying the number density of particles by the change in kinetic energy per collision and the estimated collision frequency of magnitude $\langle v^2\rangle^{\frac{1}{2}}/(2b-\sigma)$ gave the total rate of change of fluctuation energy per unit volume, which was then equated to the rate of work done by stresses and the rate of energy dissipation per unit volume. By comparing the forms of both sides, Ogawa *et al.* proposed the following constitutive equations for the stress tensor and the rate of energy dissipation γ_0:

$$\boldsymbol{p} = \frac{-\rho}{4(1-(\nu/\nu^*)^{\frac{1}{3}})}(K_1\langle v^2\rangle\,\delta_{ij}+b\langle v^2\rangle^{\frac{1}{2}}(K_2\,D_{ij}+K_3\,D_{kk}\,\delta_{ij})), \tag{3.18}$$

$$\gamma_0 = -K_0\frac{\rho}{4(1-(\nu/\nu^*)^{\frac{1}{3}})}\frac{\langle v^2\rangle^{\frac{3}{2}}}{b}, \tag{3.19}$$

where ν^* was said to correspond to the 'packed state' of granular materials, $D_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$ is the rate-of-strain tensor, and δ_{ij} is the unit tensor. In (3.18), the sign convention of Ogawa *et al.* for the stress has been changed to conform to that used here, in which compressive stresses are positive. For the case of cohesionless and smooth particles, the K_i are given as

$$K_0 = -\tfrac{1}{3}(1-e^2),\quad K_1 = -\tfrac{2}{9}e(3+e),\quad K_2 = \tfrac{1}{9}(1+e)^2, \tag{3.20a, b, c}$$

and $K_3 = 0$.

We now consider the case of simple shear flow, non-dimensionalize $\boldsymbol{p}$ and γ_0, and express them in terms of the parameter R, after noting that

$$p_{xx} = p_{yy} = p_{zz},\quad p_{yx} = p_{xy}. \tag{3.21a, b}$$

Thus

$$\frac{p_{yy}}{\rho_p\,\sigma^2(\mathrm{d}u/\mathrm{d}y)^2} = \psi_1(\nu)\frac{4e(3+e)}{9R^2}, \tag{3.22}$$

$$\frac{p_{xy}}{\rho_p\,\sigma^2(\mathrm{d}u/\mathrm{d}y)^2} = -\psi_2(\nu)\frac{(1+e)^2}{18R}, \tag{3.23}$$

$$\frac{\gamma_0}{\rho_p\,\sigma^2(\mathrm{d}u/\mathrm{d}y)^3} = \psi_3(\nu)\frac{4(1-e^2)}{3R^3}, \tag{3.24}$$

where

$$\psi_1(\nu) = \frac{1}{8}\frac{\nu}{(1-(\nu/\nu^*)^{\frac{1}{3}})}, \tag{3.25a}$$

$$\psi_2(\nu) = \psi_1\left(\frac{\nu^*}{\nu}\right)^{\frac{1}{3}},\quad \psi_3(\nu) = \psi_1\left(\frac{\nu}{\nu^*}\right)^{\frac{1}{3}}, \tag{3.25b, c}$$

and ρ_p is the mass density of an individual solid particle.

The R-dependence of the stresses $\boldsymbol{p}$ and dissipation rate γ_0 resembles the R-dependences of the first terms in the small-R solutions for $\boldsymbol{p}$ and γ given by (3.13)–(3.16). Unfortunately, the stresses of both theories depend differently on the coefficient of restitution e, and we cannot compare them directly until a relationship between e and R is established for the theory of Ogawa *et al.* However, we may

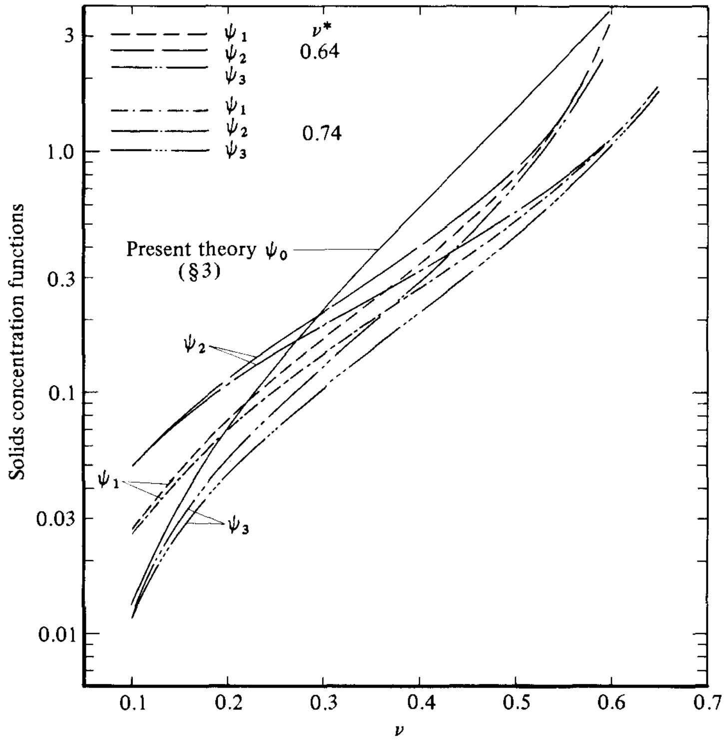

FIGURE 2. Solids concentration functions of various theories in terms of solids fraction ν.

compute the non-dimensional rate of energy dissipation if we divide (3.19) by a normalizing concentration function defined as

$$\psi_0(\nu) = \nu^2 g_0(\nu). \tag{3.26}$$

Taking the solids fraction ν to be 0.5 for the sake of comparison and $\nu^* = 0.64$ for a randomly packed mass of particles, the first term of the small-R solution of the present theory gives $\gamma^* = 1.303R^{-3}$ from (3.16), whereas the theory of Ogawa *et al.* yields

$$\gamma_0^* = \frac{\gamma_0}{\rho_p \nu^2 g_0 \sigma^2 (\mathrm{d}u/\mathrm{d}y)^3 (1-e^2)} = 0.648R^{-3}$$

(note $\rho = \nu\rho_p$). If we take the packed state in the theory of Ogawa *et al.* to correspond to a regular array of closest-packed spheres, then $\nu^* = 0.74$ and $\gamma_0^* = 0.398R^{-3}$. This comparison shows that the rate of energy dissipation of Ogawa *et al.* is about $\frac{1}{2}$ or $\frac{1}{3}$ of that of the present theory for small R, depending on how their 'packed state' is interpreted. Since they did not specify the value ν^* they had in mind, both values of ν^* of 0.64 and 0.74 will be used in their theory for the comparison that follows.

Equating the rate of work done by stresses to the energy dissipation per unit volume for the case of a steady simple shear flow, we obtained a relationship between R and e:

$$R^2 = 24\left(\frac{\nu}{\nu^*}\right)^{\frac{2}{3}} \frac{1-e}{1+e}. \tag{3.27}$$

Equation (3.27) is plotted in figure 1; the variation of R with e has the same general shape as that given by equation (3.17) of the present theory.

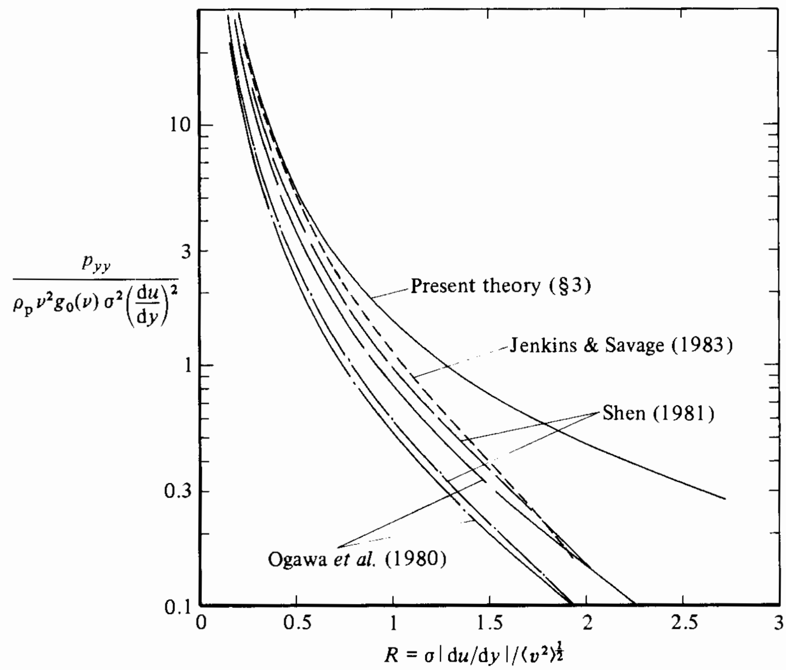

FIGURE 3. Variation of non-dimensional normal-stress component p^*_{yy} of various theories with the ratio of characteristic mean shear velocity to fluctuation velocity R for the case of simple shear. The designations of the various theories are described in the caption to figure 1.

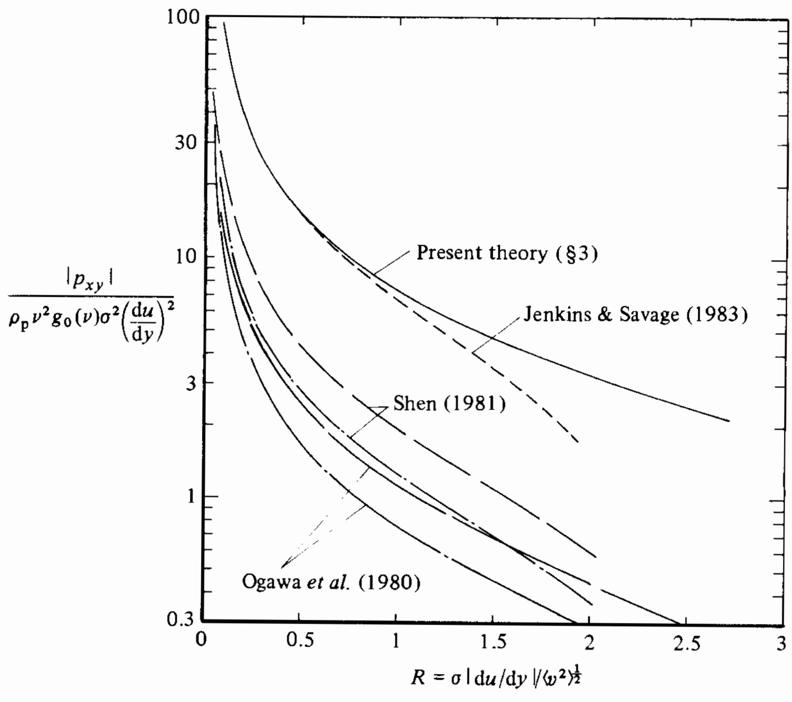

FIGURE 4. Variation of non-dimensional shear-stress component p^*_{xy} of various theories with the ratio of characteristic mean shear velocity to fluctuation velocity R for the case of simple shear. The designations of the various theories are described in the caption to figure 1.

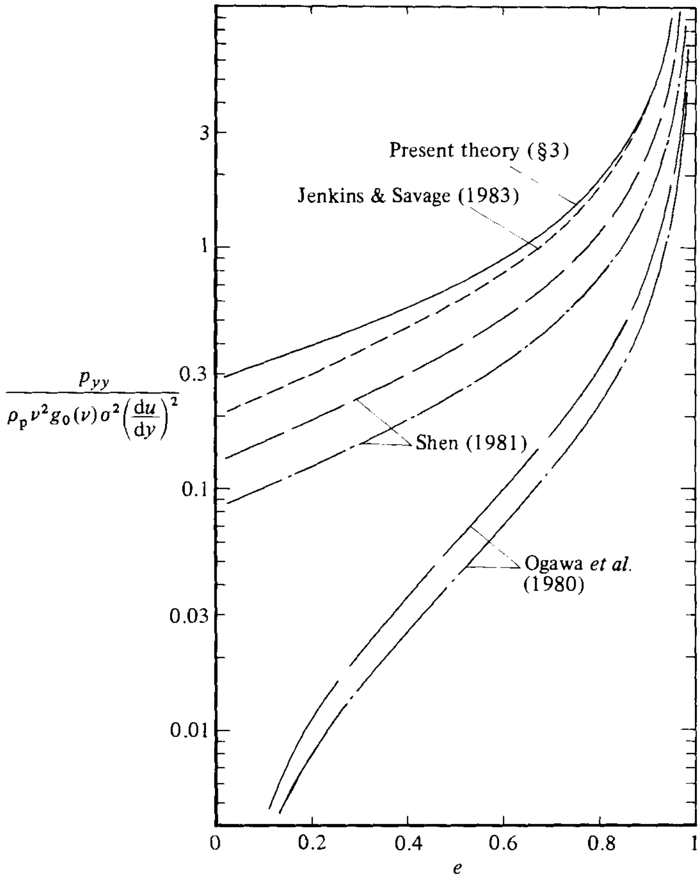

FIGURE 5. Variation of non-dimensional normal-stress component p^*_{yy} of various theories with the coefficient of restitution e. The designations of the various theories are described in the caption to figure 1.

Now we may compare the stresses by dividing (3.22) and (3.23) by (3.26):

$$\frac{p_{yy}}{\rho_p \nu^2 g_0 \sigma^2 (\mathrm{d}u/\mathrm{d}y)^2} = C_1(\nu)\frac{4e(3+e)}{9R^2}, \tag{3.28}$$

$$\frac{p_{xy}}{\rho_p \nu^2 g_0 \sigma^2 (\mathrm{d}u/\mathrm{d}y)^2} = -C_2(\nu)\frac{(1+e)^2}{18R}, \tag{3.29}$$

where

$$C_1(\nu) = \frac{\psi_1(\nu)}{\psi_0(\nu)}, \quad C_2(\nu) = \frac{\psi_2(\nu)}{\psi_0(\nu)}. \qquad (3.30), (3.31)$$

The functions ψ_1, ψ_2 and ψ_3 given by (3.25) associated with the normal stresses, shear stresses and the rate of energy dissipation per unit volume respectively are shown in figure 2 together with $\psi_0(\nu)$ of the present theory given by (3.26). The variations with concentration ν are rather different. Using the R versus e relationship of (3.27), we may determine the stresses. The variations of the normal and shear stresses in terms of R and e are shown in figures 3–6 together with those of the present theory. The stresses of Ogawa *et al.* are much lower than the present predictions.

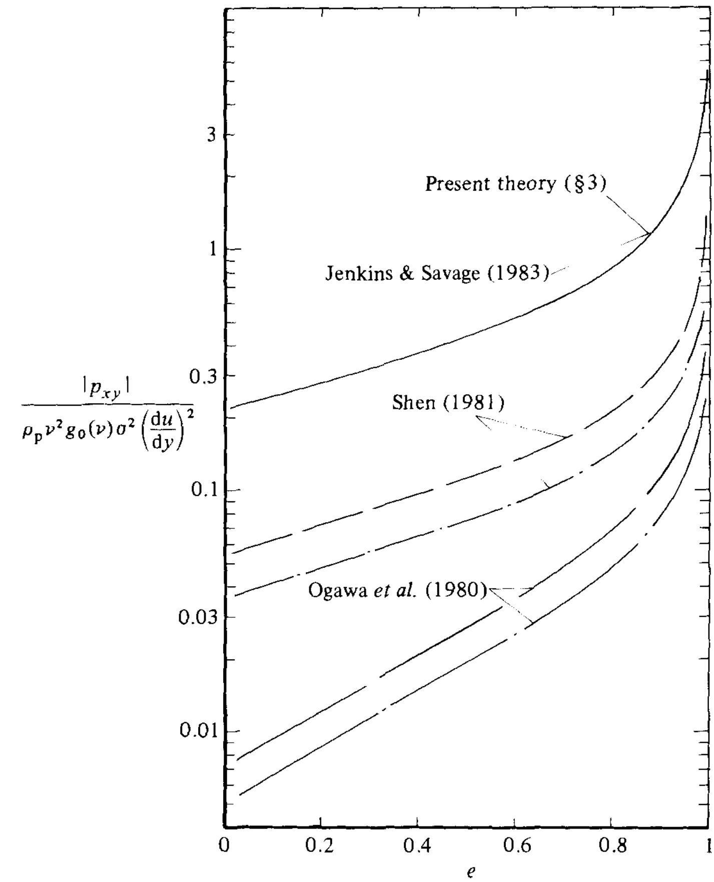

FIGURE 6. Variation of non-dimensional shear-stress component p^*_{xy} of various theories with the coefficient of restitution e. The designations of the various theories are described in the caption to figure 1.

3.3.2. *Shen* (1981)

Shen (1981) used a statistical model similar to that of Ogawa *et al.* (1980) and derived her constitutive equations for rough, inelastic spherical particles including interstitial-fluid drag effects. The case of simple shear of granular materials was considered and assumptions similar to those of Ogawa *et al.* (1980) were made. The particle velocity fluctuations were assumed to be isotropic and rotary inertia effects were neglected. The stresses were determined by letting

$$p_{ij} = p_i \Delta M_j f, \tag{3.32}$$

where p_i is the average number of particles per unit area normal to the ith coordinate direction, ΔM_j is the average jth component of the momentum transfer per collision and f is the collisional frequency of a particle inside that unit area. A similar formulation for the stresses was originally used by Bagnold (1954). The rate of energy dissipation per unit volume was given by

$$\gamma = nFE, \tag{3.33}$$

where n is the number of particles per unit volume, F is the collisional frequency of a particle and E is the energy loss of each particle per collision. The magnitude of the collision frequency F was estimated in a manner similar to that of Ogawa *et al.* (1980) by writing

$$F = 2f = \frac{\langle v^2 \rangle^{\frac{1}{2}}}{S}, \tag{3.34}$$

where S is the mean separation distance of the particles.

In order to simplify the analysis of the dynamics of particle collisions, a number of approximations were made. The main one is that $R(1+1/\lambda) \ll 1$, where $\lambda = \sigma/S$ is defined as the linear concentration of the particles. This eliminated the need to account for certain of the effects of anisotropic collisions. Another important assumption is that $\tan^{-1}\mu_p(1+e) \ll 1$, where μ_p is the coefficient of friction of the particles. For the purposes of comparison with the present theory, we consider Shen's theory for the special case of negligible interstitial-fluid effects and smooth particles. The stresses and the rate of energy dissipation then may be written in terms of the parameter R as

$$p_{xx} = p_{yy} = p_{zz}, \quad p_{xy} = p_{yx}, \tag{3.35), (3.36}$$

$$\frac{p_{yy}}{\rho_p \nu^2 g_0 \sigma^2 (du/dy)^2} = C_3(\nu) \frac{8\sqrt{2}\,(1+e)}{\pi^2 R^2}, \tag{3.37}$$

$$\frac{p_{xy}}{\rho_p \nu^2 g_0 \sigma^2 (du/dy)^2} = -C_1(\nu) \frac{0.212(1+e)}{R}, \tag{3.38}$$

$$\gamma_S^* = \frac{\gamma_S}{\rho_p \nu^2 g_0 \sigma^2 (du/dy)^3 (1-e^2)} = C_3(\nu) \frac{5}{4R^3}, \tag{3.39}$$

where

$$C_3(\nu) = \psi_3/\psi_0(\nu). \tag{3.40}$$

The above stresses and rate of energy dissipation per unit volume have the same form of R-dependence as those of Ogawa *et al.* and the first terms of the small-R expansions for $\boldsymbol{p}$ and γ of the present theory. However, the concentration functions associated with the stresses differ from those of Ogawa *et al.* A formula relating R and e may be established easily by using the balance of rate of shear work and energy dissipation

$$R^2 = \left(\frac{\nu}{\nu^*}\right)^{\frac{1}{3}} \frac{1-e}{0.212}. \tag{3.41}$$

This equation is shown together with the previous ones in figure 1; R has a 'physical' upper limit of about 2 when $e = 0$.

Computations for the stresses in terms of R and e are performed in a manner similar to those of Ogawa *et al.* (1980) and are shown in figures 3–6. The magnitudes of Shen's stresses are higher than those of Ogawa *et al.*; however, they are still considerably lower than those of the present theory.

3.3.3. *Jenkins & Savage* (1983)

Jenkins & Savage (1983) have considered the flow of smooth, nearly elastic spherical granular materials. Their theory was derived for general deformations but is an asymptotic analysis for small deformation rates (small R for the case of simple shear). Their pair-distribution function $f^{(2)}$ and single-particle distribution function $f^{(1)}$ were assumed to have the same forms used in the present theory. A collisional pair-distribution function $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ was assumed on the basis of dimensional arguments and small deformation rates to be

$$g(\boldsymbol{r}_1, \boldsymbol{r}_2) = g_0(\nu)\left(1 - \frac{\alpha\sigma \boldsymbol{kk}:\nabla \boldsymbol{u}}{(\pi T)^{\frac{1}{2}}}\right), \tag{3.42}$$

where α is some (undetermined) constant. If α is taken to be unity, their $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is essentially the first-order expansion for small R of the $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ of the present theory which was used previously by Savage & Jeffrey (1981). By using the complete pair distribution functions linearized for small deformation rates, the integrals for the collisional flux of fluctuation energy $\boldsymbol{q}_c$, the collisional stress tensor $\boldsymbol{p}_c$ and the rate of energy dissipation per unit volume γ were evaluated to yield

$$\boldsymbol{q}_c = -\kappa \nabla T, \tag{3.43}$$

$$\boldsymbol{p}_c = (\kappa\sigma^{-1}(\pi T)^{\frac{1}{2}} - \tfrac{1}{5}\kappa(2+\alpha)\,\mathrm{tr}\,\boldsymbol{D})\,\boldsymbol{I} - \frac{2\kappa}{5}(2+\alpha)\,\boldsymbol{D}, \tag{3.44}$$

$$\gamma = 6(1-e)\,\kappa\left[T - \left(\frac{\pi}{4} + \frac{\alpha}{3}\right)\sigma\left(\frac{T}{\pi}\right)^{\frac{1}{2}}\mathrm{tr}\,\boldsymbol{D}\right]\sigma^{-2} \tag{3.45}$$

where

$$\kappa = 2\nu g_0(\nu)\,(1+e)\,\rho\sigma\left(\frac{T}{\pi}\right)^{\frac{1}{2}}, \tag{3.46}$$

$$\boldsymbol{D} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

and $\boldsymbol{I}$ is the unit tensor.

Considering the case of simple shear $\boldsymbol{u} = u(y)\,\boldsymbol{e}_x$ with no fluctuation gradients, i.e. $\boldsymbol{q}_c = 0$, the constitutive equations with $\alpha = 1$ yield

$$p^*_{xx} = p^*_{yy} = p^*_{zz} = \frac{4}{3R^2}, \tag{3.47}$$

$$p^*_{xy} = p^*_{yx} = -\frac{12}{5(3\pi)^{\frac{1}{2}}R}, \tag{3.48}$$

$$\gamma^* = \frac{4}{(3\pi)^{\frac{1}{2}}R^3}, \tag{3.49}$$

which are exactly the first terms of the small-R expansions of the present theory. The relationship between the parameter R and the coefficient of restitution e is found to be

$$R^2 = \tfrac{10}{3}(1-e), \tag{3.50}$$

and is plotted along with the others in figure 1. It shows the same functional behaviour as that of Shen (1981), but its upper bound of $R = 1.83$ at $e = 0$ is lower. The non-dimensional stresses given by (3.47), (3.48) and (3.50) are plotted against both R and e in figures 3–6. In the figures showing stresses versus R, there are noticeable differences in the magnitudes of the stresses of Jenkins & Savage and those of the present theory. In the plots of stresses versus e, the magnitudes of the shear stress of Jenkins & Savage and that of the present theory are too close to be distinguished on the graph, while small differences in the normal stresses are detectable at lower e.

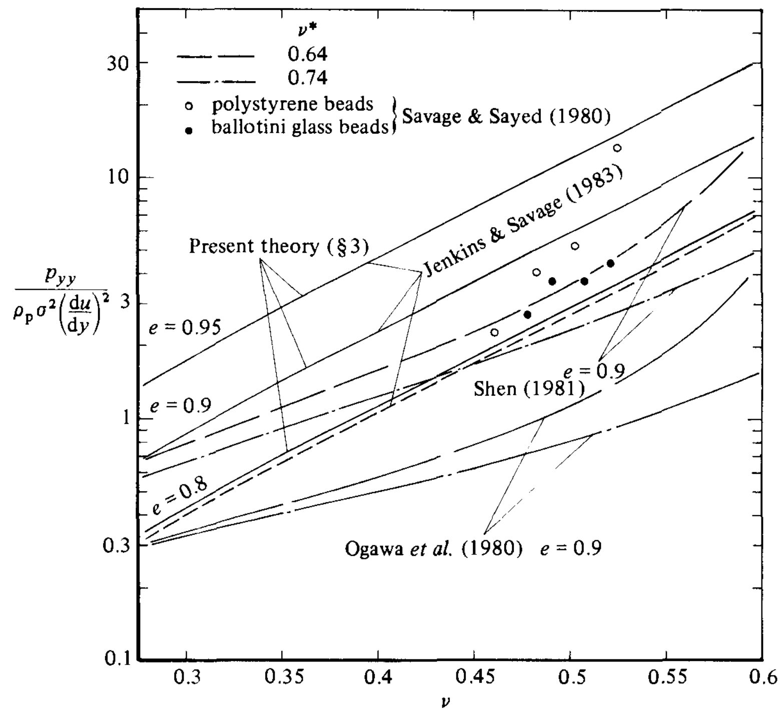

FIGURE 7. Variations of non-dimensional normal stress with solids fraction ν for the case of simple shear. Comparison of various theories with experiments of Savage & Sayed (1980).

3.3.4. *Comparison with experimental data*

Unfortunately, there are almost no experimental stress, strain-rate data that can be used to test in detail the validity of the various theories mentioned in §§3.3.1–3.3.3. In fact, the only experimental data yet presented that deal with dry granular materials sheared at high rates are the data from the annular-shear-cell tests of Savage & Sayed (1980). The range of concentration ν tested was roughly between 0.44 and 0.54, which is considered to be sufficiently high that the kinetic or diffusional contributions to the stresses are negligible. From this standpoint at least these data are appropriate for comparison with theories concerned with collisional transfer of momentum and energy. Some of the materials used were 1.0 mm diameter polystyrene spheres of specific gravity 1.095, and 1.8 mm diameter ballotini spherical glass beads of specific gravity 2.97. Data from these materials are shown in figures 7 and 8.

Values for the material properties such as coefficient of restitution must be specified in order to calculate the stresses in each of the four theories. Unfortunately, the values of e were not measured in the experiments of Savage & Sayed (1980), and we have made the theoretical calculations for a range of representative values of e. For the theories of Ogawa *et al.* (1980) and Shen (1981) with the coefficient of friction taken to be zero, only computations of $e = 0.9$ are shown, whereas, for the theory of Jenkins & Savage (1983) and the present analysis, calculations of $e = 0.95$, 0.9 and 0.8 are shown in figures 7 and 8. These values of e from 0.95 to 0.8 are probably in the appropriate range for the glass beads (see Goldsmith 1960). Little information about the value of e for the polystyrene beads is available: but it is probably lower than that for the glass beads. Since stresses increase with increasing values of e as predicted

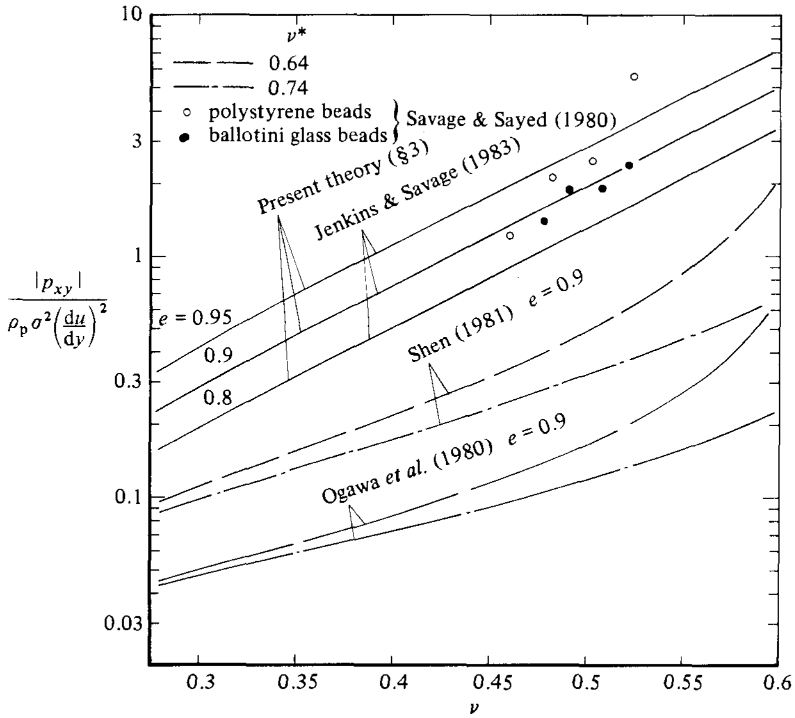

FIGURE 8. Variations of non-dimensional shear stress with solids fraction ν for the case of simple shear. Comparison of various theories with experiments of Savage & Sayed (1980).

by the present model, one might anticipate that the stresses developed by the glass beads would be higher than those developed by the polystyrene beads. However, the experimental data show the opposite trend; the stresses for the polystyrene beads are in fact somewhat higher. This difference between the predictions of the present and Jenkins & Savage analyses and experimental measurements is due to the incompleteness of these analyses in that surface friction of the granular materials has been ignored. Glass beads are brittle; when they are sheared at high shear rates under high loads, the surface of each bead gradually is roughened as a result of the multitude of collisions it experiences and the minute fractures that can occur. In such instances surface friction could become an important energy-dissipation mechanism. According to the theories of Ogawa *et al.* and Shen, which account for the particles' surface friction, the stresses decrease with increasing values of the particles' surface coefficient of friction. Thus it is possible for the stresses associated with the glass beads to be lower than those of the polystyrene beads since the coefficient of friction of the glass beads is probably higher than that of the polystyrene beads.

At $e = 0.9$ the shear stress predicted by the present theory and that of Jenkins & Savage (1983) passes through most of the experimental data, but the predicted normal stresses are somewhat higher than the measurements. Thus these two theories predict nearly the right magnitudes for the stresses. Also, it is seen that the small-R linearization is close to the more exact evaluation of the collision integrals for the higher values of e. At the same value of $e = 0.9$, the normal stress predicted by Shen is very close to the experimental data especially when $\nu^* = 0.64$, but the shear stress differs considerably. Both shear and normal stress predicted by Ogawa *et al.* have the lowest magnitude and fall short by a large amount. The theories of Ogawa *et al.* and

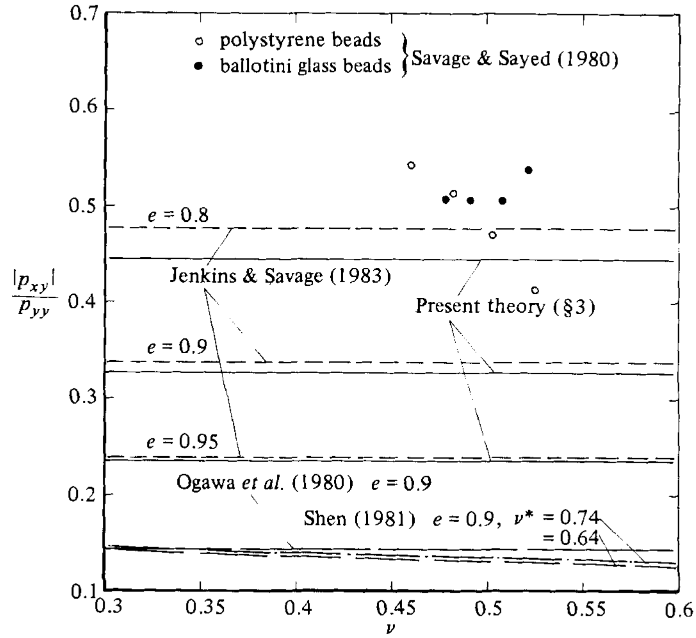

FIGURE 9. Variation of the ratio of shear stress to normal stress with solids fraction ν for the case of simple shear. Comparison of various theories with experiments of Savage & Sayed (1980).

Shen are shown for a particular coefficient of friction of zero; if $\mu_p \neq 0$ their stress predictions are reduced further.

As complementary information, we may plot the ratio of shear to normal stress against solids concentration ν as in figure 9. The stress ratio for the present theory may be found readily by dividing (3.15) by (3.13). Similarly from (3.47) and (3.48) the stress ratio from Jenkins & Savage (1983) is

$$\frac{|p_{xy}|}{p_{yy}} = 3\left[\frac{2(1-e)}{5\pi}\right]^{\frac{1}{2}}. \tag{3.51}$$

From (3.22), (3.23) and (3.27) the stress ratio corresponding to the theory of Ogawa *et al.* is

$$\frac{|p_{xy}|}{p_{yy}} = \frac{(1+e)^2}{e(3+e)}\left[\frac{3(1-e)}{8(1+e)}\right]^{\frac{1}{2}}. \tag{3.52}$$

Also from (3.37) and (3.38) the stress ratio from Shen's theory is

$$\frac{|p_{xy}|}{p_{yy}} = \frac{0.1151\pi^2}{2\sqrt{2}}\left(\frac{\nu^*}{\nu}\right)^{\frac{1}{6}}(1-e)^{\frac{1}{2}}. \tag{3.53}$$

As shown in figure 9, all the above theoretical predictions fall below the experimental values. The present analysis and that of Jenkins & Savage yield predictions closest to the test results. The shear to normal stress ratios of Ogawa *et al.* and Shen are both about $\frac{1}{4}$ of the experimental values when $e = 0.9$. One characteristic evident in figure 9 is that the stress ratios predicted by all of the theories except Shen's do not depend upon concentration ν. The experimental stress ratio for the glass beads indicates a slight dependence on concentration, while the experimental stress ratio for the polystyrene beads indicates a stronger dependence.

Finally, it should be pointed out that, in making the above comparisons with the experimental results, we have tacitly assumed that the conditions of no-slip and zero flux of fluctuation energy were achieved at the shearing walls of the test apparatus. Flow-visualization experiments in other flow situations indicate that the first assumption is reasonably accurate. The validity of the second assumption is hard to establish at the present time; a detailed analysis of wall boundary conditions analogous to the theoretical treatments described herein is required.

3.3.5. *Comparison with the Chapman–Enskog dense-gas theory*

Certain aspects of the present analysis and that of Jenkins & Savage (1983) were based upon the approach used in the Chapman–Enskog hard-sphere kinetic theory of dense gas (Chapman & Cowling 1970). The main departure initially suggested by Savage & Jeffrey (1981) is that the forms for the one- and two-particle distribution functions were merely assumed to have simple but plausible forms; they were not derived as approximate solutions to the Boltzmann equation as in the usual dense-gas approaches. It is of interest therefore to compare the present stresses with those given by the Chapman–Enskog theory. Taking only the part of the stresses that arise from the collisional transfer of momentum in correspondence to the present analysis, and expressing the results from Chapman & Cowling (1970) in the present notation, we obtain

$$\boldsymbol{p} = \left(\tfrac{4}{3}\langle v^2\rangle - \frac{5.09}{6}\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\langle v^2\rangle^{\frac{1}{2}}\sigma\boldsymbol{\nabla}\cdot\boldsymbol{u}\right)\rho\nu\chi_0(\nu)\boldsymbol{I}$$

$$-\frac{1.016}{6}\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\sigma\rho\langle v^2\rangle^{\frac{1}{2}}\left(1+\frac{38.06}{5}\nu\chi_0(\nu)\right)(\boldsymbol{D}-\tfrac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u}\boldsymbol{I}), \quad (3.54)$$

$$\chi_0(\nu) = 1+\tfrac{5}{2}\nu+4.5904\nu^2, \quad (3.56)$$

where $\chi_0(\nu)$ corresponds to the equilibrium radial distribution function at contact.

In the case of simple shear, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, and the non-dimensional normal and shear stresses become

$$\frac{p_{yy}}{\rho_p(\sigma\,\mathrm{d}u/\mathrm{d}y)^2} = \frac{4\nu^2\chi_0(\nu)}{3R^2}, \quad (3.57)$$

$$\frac{|p_{xy}|}{\rho_p(\sigma\,\mathrm{d}u/\mathrm{d}y)^2} = \frac{1.016}{12}\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\left(\nu+\frac{38.06}{5}\nu^2\chi_0(\nu)\right)\frac{1}{R}. \quad (3.58)$$

These results were derived for perfectly elastic hard spheres. To make comparisons with the present analysis we take $e = 1$ and need only consider the first-term small-R solutions; thus from (3.13) and (3.15) the normal and shear stresses are

$$\frac{p_{yy}}{\rho_p(\sigma\,\mathrm{d}u/\mathrm{d}y)^2} = \frac{4\nu^2 g_0(\nu)}{3R^2}, \quad (3.59)$$

$$\frac{p_{xy}}{\rho_p(\sigma\,\mathrm{d}u/\mathrm{d}y)^2} = -\frac{12\nu^2 g_0(\nu)}{5(3\pi)^{\frac{1}{2}}R}. \quad (3.60)$$

Comparing the expression for the stresses given by the two theories, one notes an obvious difference in the dependence upon ν arising from the difference between $\chi_0(\nu)$ and $g_0(\nu)$. These two functions are plotted in figure 10; they differ by a large amount at high concentrations. Since $g_0(\nu)$ is derived semi-empirically from computer simulations of molecular dynamics, it is expected that $g_0(\nu)$ is more accurate than $\chi_0(\nu)$, especially at higher concentrations. For the purpose of comparison, we not only

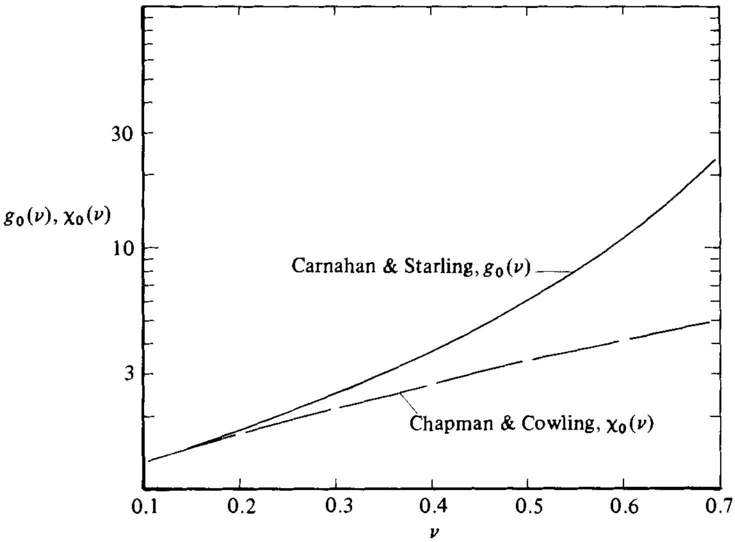

FIGURE 10. Variation of equilibrium radial distribution functions $g_0(\nu)$ and $\chi_0(\nu)$ with ν.

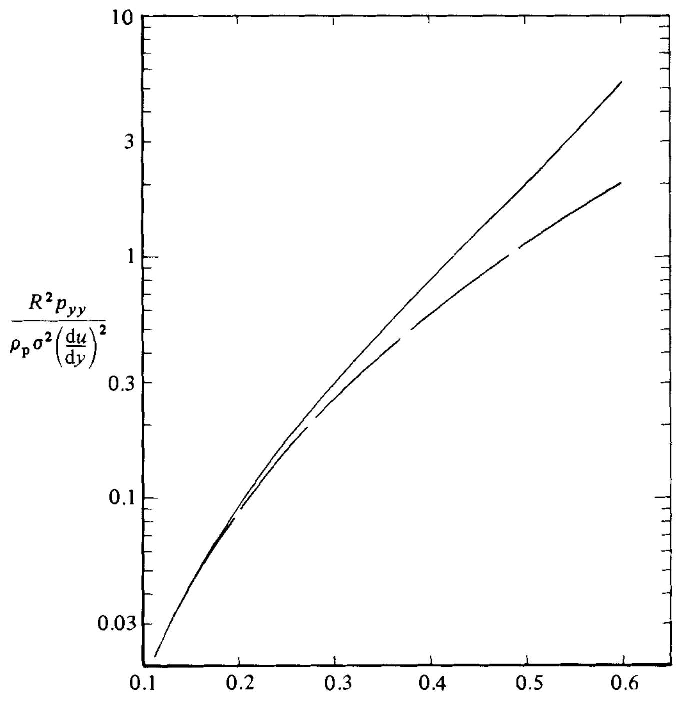

FIGURE 11. Variation of non-dimensional normal stress with solids fraction ν. Comparison of present theory with collisional contributions in Enskog dense-gas theory. — —, Chapman & Cowling (1970) using their function $\chi_0(\nu)$; ——, Chapman & Cowling (1970) replacing $\chi_0(\nu)$ with $g_0(\nu)$ and present §3 theory using $g_0(\nu)$.

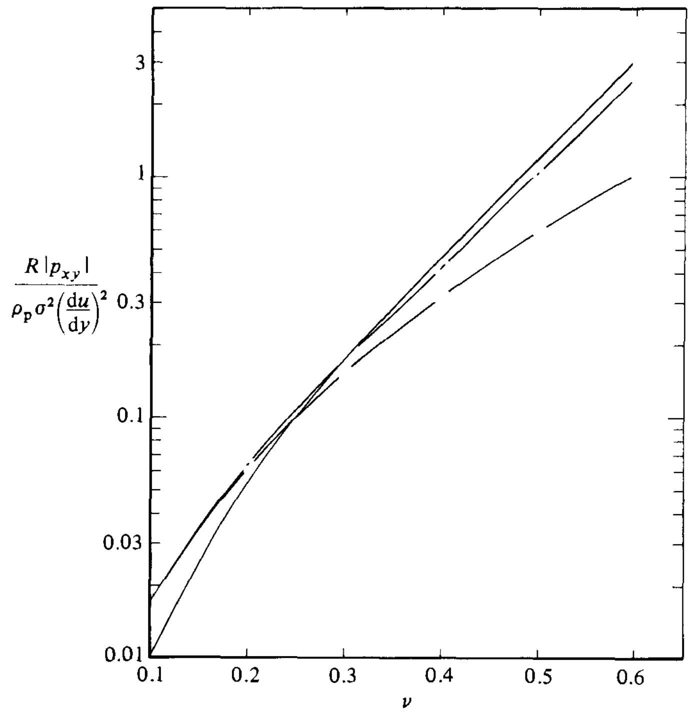

FIGURE 12. Variation of non-dimensional shear stress with solids fraction ν. Comparison of present theory with collisional contribution in Enskog dense-gas theory. — —, Chapman & Cowling (1970) using their function $\chi_0(\nu)$; ——–, Chapman & Cowling (1970) replacing $\chi_0(\nu)$ with $g_0(\nu)$; ——, present §3 theory using $g_0(\nu)$.

used the expression for $\chi_0(\nu)$ as given by Chapman & Cowling to calculate their stresses, but also we replaced $\chi_0(\nu)$ by $g_0(\nu)$ in (3.57) and (3.58) to give a closer correspondence to the present analysis. These two calculations of stresses are shown in figures 11 and 12 together with the stresses predicted by equations (3.59) and (3.60) of the present analysis. If $\chi_0(\nu)$ is used in the Chapman–Enskog theory, it is found that the normal stresses are less than those of the present theory. But, if $\chi_0(\nu)$ is replaced by $g_0(\nu)$, their shear stress is quite close to the present one, especially at concentrations of $\nu = 0.2$ and higher; the normal stresses are exactly the same.

These comparisons show that the simple approach followed by Savage & Jeffrey, Jenkins & Savage and the present paper gives results for stresses which are quite close to the more formal and rigorous analyses used in the hard-sphere dense-gas theories, at least for the case of simple shear.

3.4. *Conclusion*

As a result of the present study we may draw a number of important conclusions which concern not only the present and past theoretical work but also bear upon the direction of future theoretical studies.

1. While all of the analyses examined in §3.3 exhibit many of the same gross features of behaviour for R versus e, and for stresses versus R, e *and* ν, there are in some cases significant *quantitative* differences between them. The theories of Shen (1981) and Ogawa *et al.* (1980) are more general in that they both consider surface friction and Shen considers interstitial fluid effects, but they do predict lower stresses

than those of the present analysis and that of Jenkins & Savage (1983). The reason for this is hard to isolate, but probably it is due to the cumulative effect of a number of approximations made by Shen and Ogawa *et al.* in describing the particle kinematics and in carrying out their statistical averaging. For the case of smooth, inelastic particles all the theories yield upper bounds for the value of R at $e = 0$ ranging from about 2 to about 5.

2. The asymptotic analysis of Jenkins & Savage (1983) based upon the assumption of small gradients (which correspond to $1-e \ll 1$, and for the case of simple shear, small R) is found to yield predictions for the stresses that asymptotically approach the present analysis (which evaluates the collision integrals 'exactly') as e approaches unity and are fairly close to it for $e = 0$ and when $R > 1$. This is certainly not what one might have anticipated on the basis of the stress-versus-R curves presented in Savage & Jeffrey (1981), but it is a fortuitous result. It indicates that small-gradient, strong-fluctuation asymptotic analyses are adequate, at least for the case of theories for dry granular material. On the other hand, for cases where the interstitial fluid is important and acts to dampen the velocity fluctuations, typical values of R could well exceed the upper bounds described above (the theory of Shen (1981), which can include interstitial fluid effects clearly indicates such trends). For these situations, the asymptotic solution may well prove to be inadequate, and the inclusion of higher-order terms is probably necessary.

3. The present analysis and that of Jenkins & Savage (1983) predict stresses based upon reasonable values of e that are fairly close to the experimental stresses measured in the annular shear-cell experiments of Savage & Sayed (1980). These two analyses also predict shear stresses for perfectly elastic particles in a simple shear flow that are very close to the collisional stresses given by the Chapman–Enskog dense-gas theory, and collisional normal stresses that are the same. Making similar comparisons for more general types of flow (for example, extensional flow) suggests that the use of the simple pair distribution function proposed by Savage & Jeffrey (1981) results in constitutive equations that approximate the more exact results of the Chapman–Enskog approach to within factors of about 2.

4. Although the theory of Jenkins & Savage (1983) for general deformations is less accurate than the Chapman–Enskog approach, by comparison it is *relatively* simple and it may be expedient to extend this analysis to consider the effects of particle surface friction and rotary inertia. However, we note that the pair-distribution function of Savage & Jeffrey (1981) cannot describe the 'kinetic' or 'diffusional' contributions of the stresses that can be important at low concentrations.

4. Hard-sphere, dense-fluid kinetic theory for nearly elastic particles

The comparisons of the various analyses and experiments made in §3 suggest the development of a more detailed theory following the approach of either the Chapman–Enskog or the moment methods, which are based on the assumption (explicit or implicit) that gradients of the mean-flow properties such as velocity, temperature and bulk density are in some sense small. Such an analysis is now presented. As we have seen in §3, the small-gradient assumption in the present context implies that the energy dissipated during a collision is small, and hence that the smooth particles considered here are nearly elastic (the coefficient of restitution e is nearly unity). We expect the analysis for inelastic particles to reduce to the standard Enskog results for dense hard-sphere fluids (Chapman & Cowling 1970; Ferziger & Kaper 1972; Hirschfelder, Curtis & Bird 1954; Davis 1973) in the limit as $e \to 1$. The resulting constitutive equations are bound to be fairly complex, and their use will present difficulties in the solution of any but the most simple of flow problems. Nevertheless, these constitutive equations, which are calculated by using distribution functions which have been determined in a systematic way. provide means to assess the accuracy of simpler, but more approximate, *ad hoc* analyses such as those discussed in §3 and in Jenkins & Savage (1983).

4.1. *Analysis*

Rather than use the machinery described by Chapman & Cowling (1970), which relies upon the Boltzmann equation, we use a simple moment method based on the Maxwell transport equation (2.2) and (2.7)–(2.9). We follow the spirit of the approach used in the early work of Chapman (1916), which has been described in elementary terms for the dilute-gas case in the books of Present (1958), Guggenheim (1960) and Reif (1965). A related discussion of the relation between the Chapman–Cowling and Enskog procedure and a variational procedure has been given by Hirschfelder *et al.* (1965). The results obtained for the *perfectly elastic* hard-sphere fluids using these approaches in the first approximation correspond to keeping the first terms in an infinite Sonine polynomial expansion for the singlet perturbation in the Chapman–Enskog theory. The remaining terms for these fluids have been found to alter the calculated numerical values of the viscosity coefficients and thermal conductivity by only a few percent. We anticipate similar effects for the slightly inelastic particles treated here.

While the moment method applied here for the case of a dense system is conceptually quite straightforward, the detailed computations are lengthy and the likelihood of an algebraic error is not insignificant. The algebraic manipulations and integrations were done initially by hand. A program written in PL/1-FORMAC was developed to check the hand calculations. As a further check, the present results are compared with the standard hard-sphere dense-fluid results as $e \to 1$.

As usual we write the singlet distribution function $f^{(1)}$ in the form

$$f^{(1)} = f^{(0)}(1+\phi), \tag{4.1}$$

where $\phi \ll 1$, and

$$f^{(0)}(\boldsymbol{r}, \boldsymbol{c}, t) = \frac{n}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(\boldsymbol{c}-\boldsymbol{u})^2}{2T}\right) \tag{4.2}$$

is the *local* equilibrium Maxwellian single-particle velocity distribution function. From the standard kinetic theories we anticipate that a good approximation to the perturbation ϕ is the trial function

$$\phi = a_1 \overset{0}{\boldsymbol{CC}} : \nabla \boldsymbol{u} + a_2 \left(\frac{5}{2} - \frac{C^2}{2T}\right) \boldsymbol{C} \cdot \nabla \ln T + a_3 \left(\frac{5}{2} - \frac{C^2}{2T}\right) \boldsymbol{C} \cdot \nabla \ln n, \tag{4.3}$$

where the traceless dyadic

$$\overset{0}{\boldsymbol{CC}} = \boldsymbol{CC} - \tfrac{1}{3}C^2 \boldsymbol{I} \tag{4.4}$$

and

$$\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{u}.$$

Note that we have reverted to the use of $\boldsymbol{C}$ to denote the peculiar velocity to facilitate comparison with the standard dense-fluid theories.

The quantities a_1, a_2 and a_3 appearing in (4.3) are found by satisfying various moment equations generated by (2.2) and (2.7)–(2.9).

The first two terms in (4.3) are of the standard form, but the last term is not present in the usual theories. The necessity for the third term in (4.3) becomes apparent when attempting to satisfy the moment equations. It occurs here because of our consideration of inelastic collisions, and it will be found that, for $e = 1$, $a_3 = 0$, as might be expected. The functional dependence of this third term upon $\boldsymbol{C}$ is evident from the scalar nature of ϕ and the necessary compatibility with definition for the mean velocity $\boldsymbol{u} = \langle \boldsymbol{c} \rangle$.

We make the Enskog assumption that the pair-distribution function

$$f^{(2)}(\boldsymbol{r}_1, \boldsymbol{c}_1; \boldsymbol{r}_2, \boldsymbol{c}_2; t) = g_0(\sigma) f^{(1)}(\boldsymbol{r} - \sigma\boldsymbol{k}, \boldsymbol{c}_1; t) f^{(1)}(\boldsymbol{r}, \boldsymbol{c}_2; t) \tag{4.5}$$

just prior to collision, and take the equilibrium radial distribution function $g_0(\sigma)$ to be that due to Carnahan & Starling (1969) given by (3.3).

The coefficient a_1 appearing in (4.3) may be determined by considering the case of $\nabla\boldsymbol{u} = \boldsymbol{e}_y \boldsymbol{e}_x \mathrm{d}u/\mathrm{d}y$, $\mathrm{d}u/\mathrm{d}y = \text{const}$, $T = \text{const}$, and $n = \text{const}$ (where $\boldsymbol{u} = (u, v, w)$) and taking $\psi = c_x c_y$ in (2.2) and (2.7)–(2.9). Similarly a_2 and a_3 may be determined by considering $\boldsymbol{u} = 0$, and T and n to have gradients in the x-direction and taking $\psi = c^2 c_x$. Thus we find

$$a_1 = -\frac{\mu_\mathrm{i}}{\rho T^2 g_0}[1 + \tfrac{8}{5}\eta(3\eta - 2)\nu g_0], \tag{4.6}$$

$$a_2 = \frac{2\lambda_\mathrm{i}}{5\rho T g_0}\left[1 + \frac{12\eta^2}{5}(4\eta - 3)\nu g_0\right], \tag{4.7}$$

$$a_3 = \frac{24\lambda_\mathrm{i}\,\eta}{25\rho T g_0}(2\eta - 1)(\eta - 1)\frac{\mathrm{d}}{\mathrm{d}\nu}(\nu^2 g_0), \tag{4.8}$$

where

$$\mu_\mathrm{i} = \frac{\mu}{\eta(2-\eta)}, \quad \lambda_\mathrm{i} = \frac{8\lambda}{\eta(41 - 33\eta)}, \tag{4.9), (4.10}$$

$$\eta = \tfrac{1}{2}(1+e), \quad \mu = \frac{5m(T/\pi)^{\frac{1}{2}}}{16\sigma^2}, \quad \lambda = \frac{75m(T/\pi)^{\frac{1}{2}}}{64\sigma^2}. \tag{4.11), (4.12), (4.13}$$

The expressions for μ and λ may be identified as the shear viscosity and thermal conductivity for perfectly elastic particles ($e = 1$, $\eta = 1$) at *dilute concentrations*. They check with the usual first approximation results from the hard-sphere kinetic theory (see Davis 1973). The symbols μ_i and λ_i denote the corresponding shear viscosity and granular thermal conductivity for the case of *inelastic* particles at dilute concentrations.

Using (4.3)–(4.13) in (2.8), (2.9) and (2.16)–(2.19) we can calculate the constitutive equations for stress, translational fluctuation-energy flux and collisional rate of dissipation per unit volume. The kinetic and collisional contributions to the total stress tensor $\boldsymbol{p}$ are

$$\boldsymbol{p}_k = \rho T \boldsymbol{I} - \frac{2\mu}{\eta(2-\eta) g_0}[1 + \tfrac{8}{5}\eta(3\eta - 2)\nu g_0]\boldsymbol{S}, \tag{4.14}$$

$$\boldsymbol{p}_\mathrm{c} = 4\rho T \eta \nu g_0 \boldsymbol{I} - \frac{16\mu\nu}{5(2-\eta)}[1 + \tfrac{8}{5}\eta(3\eta - 2)\nu g_0]\boldsymbol{S} - \frac{256}{5\pi}\eta\mu\nu^2 g_0[\tfrac{6}{5}\boldsymbol{S} + (\nabla\cdot\boldsymbol{u})\boldsymbol{I}], \tag{4.15}$$

where

$$\boldsymbol{S} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) - \tfrac{1}{3}u_{k,k}\,\delta_{ij}. \tag{4.16}$$

The total stress tensor may be written as

$$\boldsymbol{p} = \boldsymbol{p}_{\mathrm{k}}+\boldsymbol{p}_{\mathrm{c}} = (\rho T(1+4\eta\nu g_0)-\eta\mu_{\mathrm{b}}\nabla\cdot\boldsymbol{u}]\boldsymbol{I}$$

$$-\left\{\frac{2\mu}{\eta(2-\eta)g_0}(1+\tfrac{8}{5}\eta\nu g_0)[1+\tfrac{8}{5}\eta(3\eta-2)\nu g_0]+\tfrac{6}{5}\mu_b\eta\right\}\boldsymbol{S}, \quad (4.17)$$

where

$$\mu_{\mathrm{b}} = \frac{256\mu\nu^2 g_0}{5\pi} \quad (4.18)$$

is the bulk viscosity for perfectly elastic particles and $\eta\mu_{\mathrm{b}}$ is that property for inelastic particles.

The kinetic and collisional contributions to the vector flux of translational fluctuation kinetic energy are

$$\boldsymbol{q}_{\mathrm{k}} = -\frac{\lambda_{\mathrm{i}}}{g_0}\left\{[1+\tfrac{12}{5}\eta^2(4\eta-3)\nu g_0]\nabla T+\tfrac{12}{5}\eta(2\eta-1)(\eta-1)\frac{\mathrm{d}}{\mathrm{d}\nu}(\nu^2 g_0)\frac{T}{n}\nabla n\right\}, \quad (4.19)$$

$$\boldsymbol{q}_{\mathrm{c}} = -\frac{12\eta\lambda_{\mathrm{i}}\nu}{5}\left\{[1+\tfrac{12}{5}\eta^2(4\eta-3)\nu g_0+\frac{16}{15\pi}(41-33\eta)\eta\nu g_0]\nabla T\right.$$

$$\left.+\tfrac{12}{5}\eta(2\eta-1)(\eta-1)\frac{\mathrm{d}}{\mathrm{d}\nu}(\nu^2 g_0)\frac{T}{n}\nabla n\right\}, \quad (4.20)$$

and the total 'thermal' flux vector is

$$\boldsymbol{q} = \boldsymbol{q}_{\mathrm{k}}+\boldsymbol{q}_{\mathrm{c}} = -\frac{\lambda_{\mathrm{i}}}{g_0}\left\{(1+\tfrac{12}{5}\eta\nu g_0)[1+\tfrac{12}{5}\eta^2(4\eta-3)\nu g_0]+\frac{64}{25\pi}(41-33\eta)(\eta\nu g_0)^2\right\}\nabla T$$

$$-\frac{\lambda_{\mathrm{i}}}{g_0}(1+\tfrac{12}{5}\eta\nu g_0)\tfrac{12}{5}\eta(2\eta-1)(\eta-1)\frac{\mathrm{d}}{\mathrm{d}\nu}(\nu^2 g_0)\frac{T}{n}\nabla n. \quad (4.21)$$

Since $\nu = \frac{1}{6}\pi n\sigma^3$, then

$$\frac{1}{n}\nabla n = \frac{1}{\nu}\nabla\nu. \quad (4.22)$$

When we consider the limit $e\rightarrow 1$, $\eta\rightarrow 1$ corresponding to perfectly elastic particles, (4.14)–(4.21) reduce to the classical results for dense hard-sphere fluids corresponding to the first terms in the Sonine polynomial expansions (Chapman & Cowling 1970; Ferziger & Kaper 1972; Hirschfelder *et al.* 1954; Davis 1973).

The expression for energy flux contains a term proportional to $\nabla\nu$. Similar 'anomalous' terms dependent upon density gradient have been noted previously in dense-gas kinetic theories and have been discussed by Dahler & Theodosopulu (1975). It should be noticed, however, that the term also contains the factor $\eta-1$ which has been assumed to be small.

The analysis has been based on the assumption of small mean-field gradients, which implies small inelasticity. The expression for the collisional rate of energy dissipation per unit volume carried out to the same order of approximation as the expressions for stress and energy flux is

$$\gamma = \frac{48}{\pi^{\frac{1}{2}}}\eta(1-\eta)\frac{\rho_{\mathrm{p}}\nu^2}{\sigma}g_0 T^{\frac{3}{2}}, \quad (4.23)$$

which vanishes as $e\rightarrow 1$. In (4.23) the ordering of $1-e$ and the non-dimensional gradients of mean-flow properties (analogous to R) has been chosen to be consistent with the simple shear flow as in (3.1).

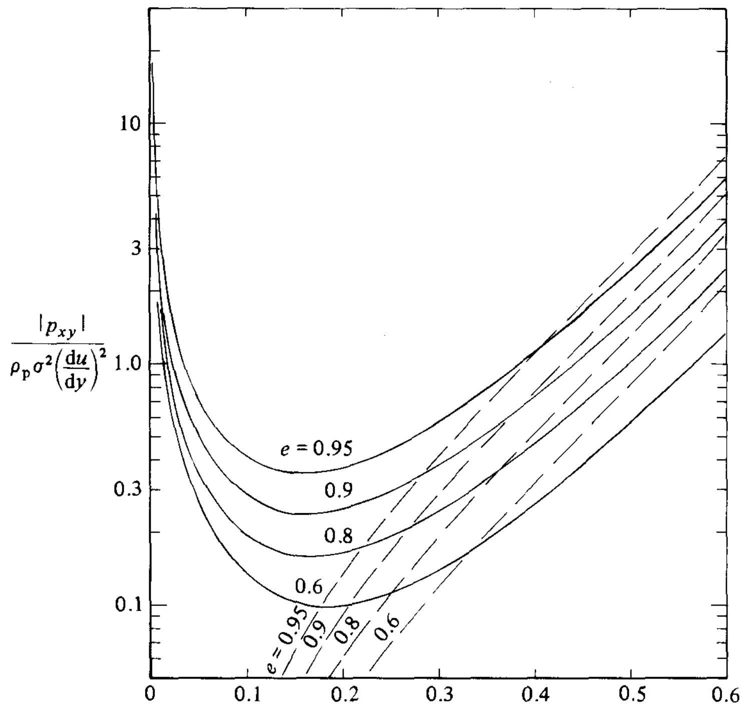

FIGURE 13. Non-dimensional shear stress versus solids fraction for case of simple shear. ——, present theory, §4; ——, Jenkins & Savage (1983).

4.2. *Simple shear flow*

It is interesting to compare the predictions of the above theory for the case of simple shear flow $\mathrm{d}u/\mathrm{d}y$, having uniform translational granular temperature T and bulk density ρ, with the predictions of the simple *ad hoc* approach described in §3. The analysis just developed incorporates the kinetic contributions to the stresses. Because of the form chosen for $f^{(2)}$ in the approach of §3, it was not possible to determine the kinetic stresses in that analysis. It transpires that the presence of the kinetic stresses results in some interesting phenomena. Following the procedure used in §3, the fluctuation-energy equation reduces to (3.1) and we obtain the following expressions for the shear and normal stresses, R and the ratio of shear to normal stress:

$$p_{xy} = p_{yx} = -F(\nu, e)\tfrac{5}{96}\rho_p (\pi T)^{\frac{1}{2}} \sigma \frac{\mathrm{d}u}{\mathrm{d}y}, \tag{4.24}$$

$$p_{yy} = p_{xx} = p_{zz} = \rho T(1 + 4\eta\nu g_0), \tag{4.25}$$

$$R = \frac{\sigma|\mathrm{d}u/\mathrm{d}y|}{(3T)^{\frac{1}{2}}} = \left[\frac{1536}{5\pi}\frac{\eta(1-\eta)\,\nu^2 g_0}{F(\nu, e)}\right]^{\frac{1}{2}}, \tag{4.26}$$

$$\frac{|p_{xy}|}{p_{yy}} = \tan\phi_D = \frac{[\frac{5}{2}\eta(1-\eta)\,g_0\,F(\nu, e)]^{\frac{1}{2}}}{1 + 4\eta\nu g_0}, \tag{4.27}$$

where

$$F(\nu, e) = \frac{1}{\eta(2-\eta)\,g_0}(1 + \tfrac{8}{5}\eta\nu g_0)\,[1 + \tfrac{8}{5}\eta\nu g_0(3\eta - 2)] + \frac{768}{25\pi}\eta\nu^2 g_0. \tag{4.28}$$

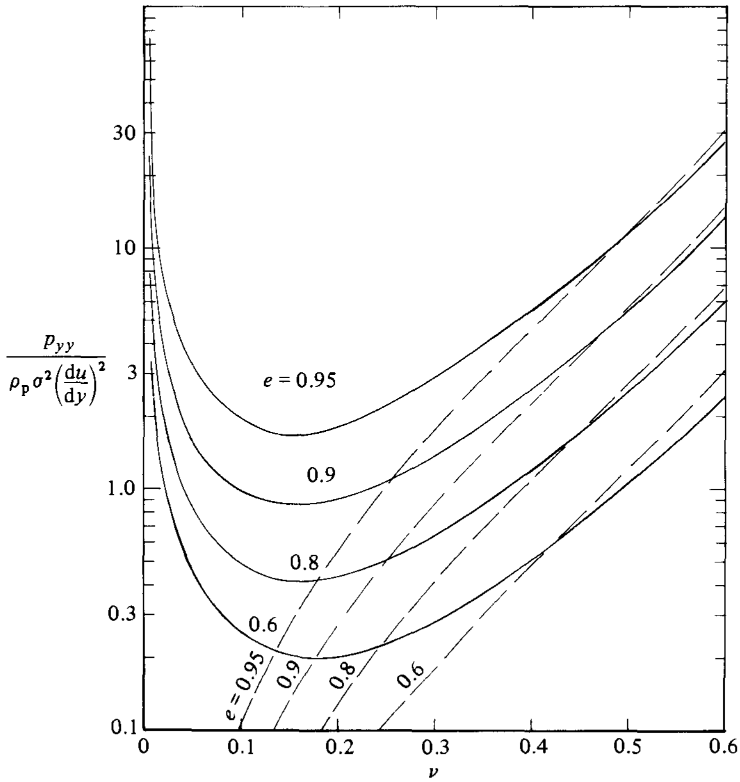

FIGURE 14. Non-dimensional normal stress versus solids fraction for the case of simple shear. ——, present theory, §4; — —, Jenkins & Savage (1983).

4.2.1. *Dependence of stresses upon* ν

Figures 13 and 14 compare the non-dimensional stresses $|p_{xy}|/[\rho_p \sigma^2(du/dy)^2]$ and $p_{yy}/[\rho_p \sigma^2(du/dy)^2]$ versus ν predicted by the above analysis with the corresponding predictions of Jenkins & Savage (1983) and the analysis of §3.1 which were based upon the *ad hoc* assumption for the pair distribution function $f^{(2)}$ and which consisted of collisional stresses only. When only the collisional-stress contributions are considered, the stresses increase in monotonic fashion with concentration. When the kinetic-stress contributions are included as in the present §4 analysis, the stresses have ***high*** values at low concentration. The stresses *decrease* with increasing concentration to a minimum, and thereafter increase with increasing concentration and follow the same trends as the collisional stresses alone. This kind of behaviour at low ν may seem implausible, but it can be explained simply in physical terms as follows. As ν decreases, the contributions of the collisional stresses becomes a smaller fraction of the total stresses and the kinetic contribution becomes dominant. Consider the shear rate du/dy to be fixed as ν decreases. From the usual arguments for Maxwell's law for a dilute gas (Jeans 1967, p. 164), the shear viscosity and thus the shear stress will be independent of density. However, the dissipation γ, which is due to collisions, is seen from (4.23) to depend upon bulk density ρ (or solids fraction ν). The dissipation γ has a stronger dependence on granular temperature T than does the shear stress. Therefore, when ν is decreased, T must increase to provide the dissipation required for the energy balance (3.1). Thus the increase in T causes the shear stress to increase as ν is decreased while keeping shear rate fixed. Note that in figures 13 and 14 the stresses increase without limit as the concentrations $\nu \to 0$. This defect is a result of the assumption of a constant coefficient of restitution in the model.

4.2.2. *An instability mechanism*

The simple shear flow just considered is unstable for small values of ν where $\partial p_{yy}/\partial \nu < 0$. If $\partial p_{yy}/\partial \nu < 0$ a local perturbation which decreases ν will cause an increase in the dispersive pressure p_{yy}, which will cause the material to dilate even further. This physical argument may be supported by a simple analysis. Consider a perturbation to the simple shear flow such that the mean-flow variables have the forms

$$\rho(y,t) = \rho_{\mathrm{p}}\, \nu(y,t) = \rho_0 + \rho_p\, \nu'(y,t), \tag{4.29}$$

$$u(y,t) = u_0(y) + u'(y,t), \tag{4.30}$$

$$v(y,t) = 0 + v'(y,t), \tag{4.31}$$

$$T(y,t) = T_0 + T'(y,t), \tag{4.32}$$

where the basic flow density ρ_0 and granular temperature T_0 are constants and the primed quantities are small perturbations which are functions only of y and t. Equations (4.29)–(4.32) are substituted into the mass-, momentum- and energy-conservation equations (2.13)–(2.15). For the present purposes we need only consider the first-order equations for conservation of mass and linear momentum in the y-direction:

$$\rho_{\mathrm{p}} \frac{\partial \nu'}{\partial t} = \rho_0 \frac{\partial v'}{\partial y}, \tag{4.33}$$

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial p_{yy}}{\partial \nu} \frac{\partial \nu'}{\partial y}. \tag{4.34}$$

To first order we take

$$\beta = \frac{1}{\rho_{\mathrm{p}}} \frac{\partial p_{yy}}{\partial \nu} = \text{const} \tag{4.35}$$

and eliminate v' from (4.33) and (4.34) to obtain

$$\frac{\partial^2 \nu'}{\partial t^2} = -\beta \frac{\partial^2 \nu'}{\partial y^2}. \tag{4.36}$$

Writing

$$\nu' = \epsilon\, \mathrm{e}^{\mathrm{i}(ky-\omega t)}, \tag{4.37}$$

we find the flow is stable or unstable depending upon whether $\beta = (1/\rho_{\mathrm{p}})\, \partial p_{yy}/\partial \nu$ is positive or negative.

This result suggests that computer simulations of granular Couette flows of the kind performed by Campbell & Brennen (1982) may never converge to a steady state for values of ν less than some critical value if e is assumed constant. In fact these kinds of stability problems have been alluded to by Campbell (1982). In his numerical simulations of flows of disks down inclined chutes, under certain conditions he was able to get flow convergence only by introducing an e that was an exponentially decaying function of particle impact velocity rather than taking it to be a constant as has been assumed in the present paper. This kind of dependence is consistent with the experimental data collected in Goldsmith (1960) and with the physical picture of how e varies with the extent of plastic deformation incurred during a collision.

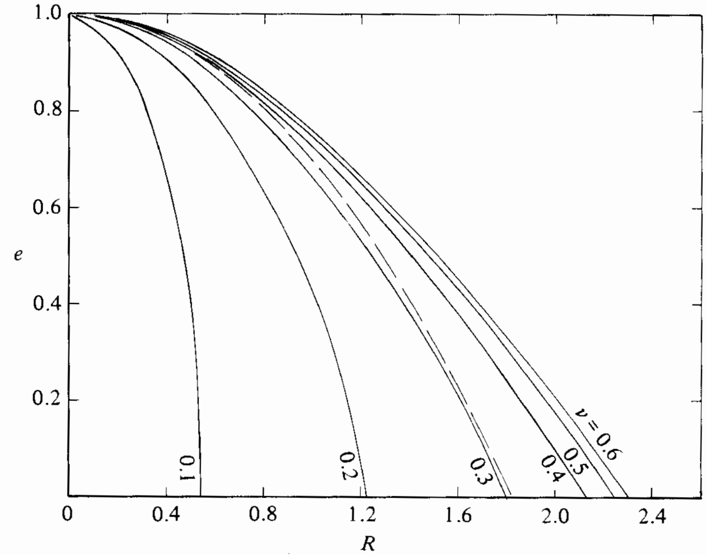

FIGURE 15. Variation of characteristic mean shear to fluctuation velocity ratio R with coefficient of restitution e for the case of simple shear. ——, present theory, §4; — —, Jenkins & Savage (1983).
(1983).

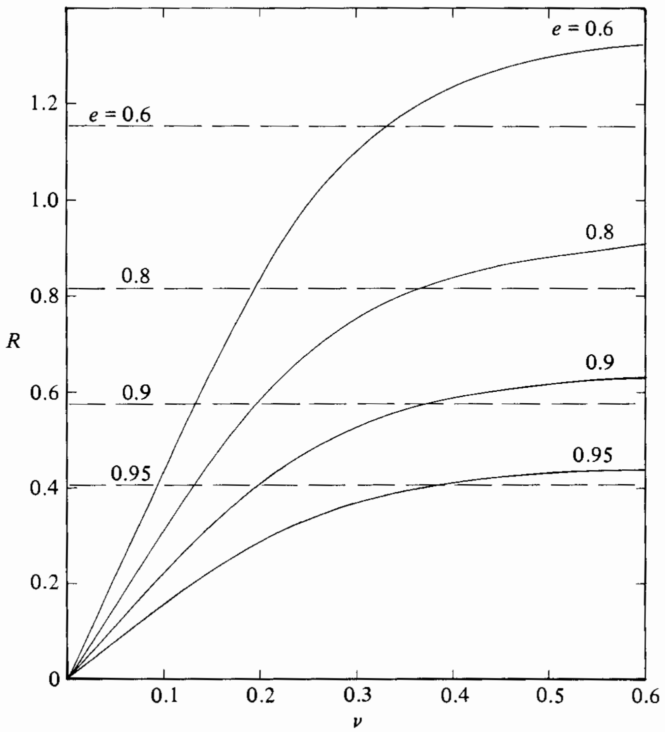

FIGURE 16. Variation of characteristic mean shear to fluctuation velocity ratio R with coefficient of restitution e for the case of simple shear. ——, present theory, §4; — —, Jenkins & Savage (1983).

4.2.3. *Variation of R and dynamic friction angle ϕ_D with ν and e*

Figures 15, 16 and 17 show the parameter R plotted versus ν, e versus R, and the ratio of shear stress to normal stress $|p_{xy}|/p_{yy} = \tan\phi_D$ plotted versus ν. The analyses of §3.1 and of Jenkins & Savage (1983) yield expressions for R and ϕ_D that depend on e but are *independent* of ν. The present analysis exhibits a ν-dependence

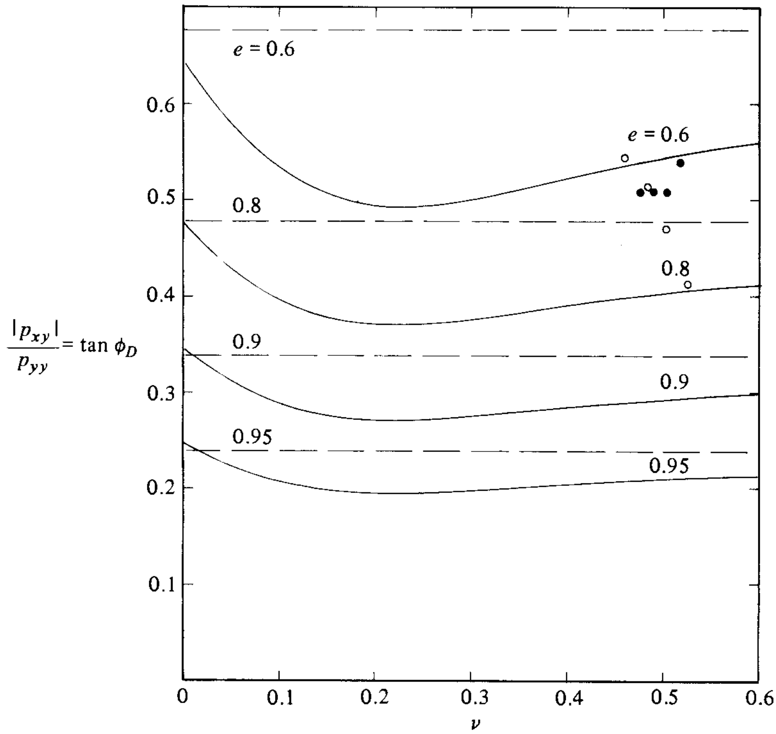

FIGURE 17. Ratio of shear to normal stress $\tan \varphi_D$ versus solids fraction ν for the case of simple shear. ——, present theory, §4; — —, Jenkins & Savage 1983; ●, glass beads; ○, polystyrene beads (Savage & Sayed 1980).

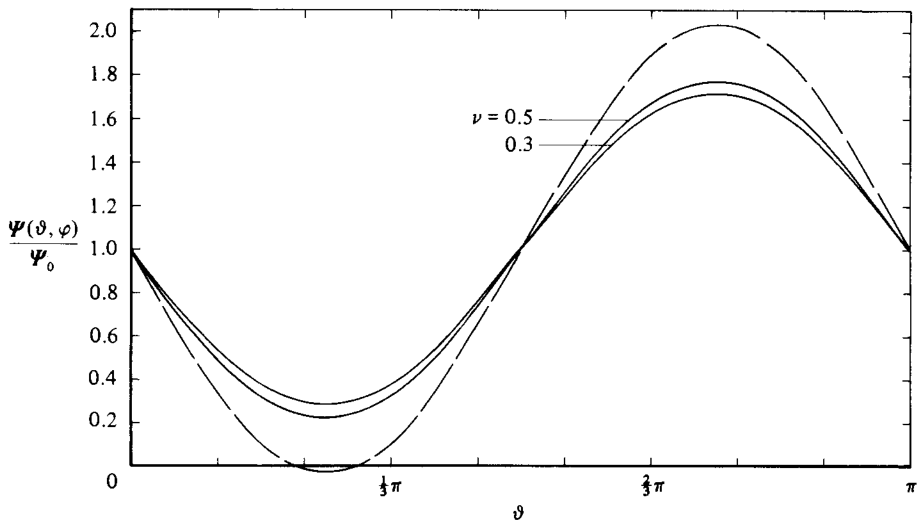

FIGURE 18. Variation of normalized collisional distribution function with angle ϑ for $e = 0.8$ and $\varphi = \frac{1}{2}\pi$. ——, present theory, §4; — —, Jenkins & Savage (1983).

for both R and ϕ_D. The curves relating e and R shown in figure 15 are similar to those corresponding to all the other theories discussed in §3. The curves of R versus ν are similar to those obtained by Campbell & Brennen (1982) in their computer simulations of idealized disk-like granular materials which showed an increase in R with increasing ν and a decrease in R with increasing e.

The ratio of shear to normal stress was found in the computer simulations of Campbell & Brennen to decrease with increasing ν and increasing e. The results of the shear-cell experiments of Savage & Sayed (1980) also exhibited a decrease in $\tan\phi_{\mathrm{D}}$ with increasing ν. The present predictions show a behaviour similar to the simulations of Campbell & Brennen for small ν, but they show an increase in $\tan\phi_{\mathrm{d}}$ with increasing ν for $\nu > 0.3$. The reason for the discrepancy is no doubt a failure of the present model at large ν. Campbell & Brennen observed that as concentration was increased there was a greater tendency for the particles to line up in distinct shear layers. Collisions tend to become restricted to (*a*) glancing collisions between particles in adjacent layers where the 'tops' and 'bottoms' of particles hit, and (*b*) collisions between particles in the same layer involving the 'fronts' and 'backs' of particles. These kinds of collisions are ineffective in developing shear stresses, but can develop normal stresses perpendicular to the shear planes. Thus one would expect that the ratio of shear stress to normal stress would decrease with increasing ν. The present theoretical model cannot produce the spike-like collisional distribution functions caused by the layering observed by Campbell & Brennen.

4.2.4. *Collisional frequency distribution*

For the present case of simple shear flow, it is interesting to examine how the collisional frequency per unit area is distributed over the collision sphere. Following Chapman & Cowling (1970, §5.2), we may express this in integral form as

$$\Psi(\vartheta,\varphi) = \sigma^3 \int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0} (\boldsymbol{c}_{12}\cdot\boldsymbol{k}) f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1;\boldsymbol{r},\boldsymbol{c}_2;t)\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{4.38}$$

By using the pair velocity distribution function given by (4.1)–(4.6), we find

$$\Psi(\vartheta,\varphi) = \frac{\sigma^3 n^2 T^{\frac{1}{2}} g_0}{\pi^{\frac{1}{2}}}\left[1-\left(1-\frac{2a_1 T^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\right)\frac{\pi^{\frac{1}{2}}\sigma}{2T^{\frac{1}{2}}}\frac{\mathrm{d}u}{\mathrm{d}y}\sin\vartheta\cos\vartheta\sin\varphi\right]. \tag{4.39}$$

For the equilibrium case (in the absence of shear), the collision frequency would be isotropic and of the form

$$\Psi_0 = \frac{\sigma^3 n^2 T^{\frac{1}{2}} g_0}{\pi^{\frac{1}{2}}}. \tag{4.40}$$

Dividing (4.39) by (4.40), we obtain the normalized collision frequency

$$\frac{\Psi(\vartheta,\varphi)}{\Psi_0} = 1-\left(1-\frac{2a_1 T^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\right)\frac{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}{2}R\sin\vartheta\cos\vartheta\sin\varphi. \tag{4.41}$$

In a similar way we obtain the normalized collision frequency from the theory of Jenkins & Savage (1983) to be

$$\frac{\Psi(\vartheta,\varphi)}{\Psi_0} = 1-\left(1+\frac{2\alpha}{\pi}\right)\frac{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}{2}R\sin\vartheta\cos\vartheta\sin\varphi. \tag{4.42}$$

Comparing (4.41) and (4.42), one sees that the normalized collisional frequency distribution of the present theory depends upon the solids concentration ν through the function a_1 (see (4.6)), whereas that given by the theory of Jenkins & Savage is independent of ν. Equations (4.41) and (4.42) are plotted in figure 18 for the case of $e = 0.8$ with the angle φ taken to be $\frac{1}{2}\pi$, i.e. we have shown collisions occurring in the plane perpendicular to the shear surfaces and parallel to the mean flow $\boldsymbol{u}$. In the expression (4.42) from the Jenkins & Savage theory, α was chosen to be unity. The collisional anisotropies from both theories have a sinusoidal dependence upon ϑ, indicating that collisions are more frequent on the 'upstream' quadrant than on the 'downstream' quadrant of the collision sphere. The anisotropy of collisions predicted by the theory of Jenkins & Savage is more pronounced than that given by the present theory; this results in collisional shear stresses that are slightly higher than the total (collisional plus kinetic) shear stresses predicted by the present theory shown in figure 13. Fortuitously, because of the form chosen by Jenkins & Savage (1983) for their pair-distribution function $f^{(2)}$, the differences in the stresses are not as great as might be expected from the comparison of the collisional frequency distribution functions. It is also worth noting that the collisional frequency distribution function based upon the pair velocity distribution function of Jenkins & Savage becomes negative near $\vartheta = \frac{1}{4}\pi$ for $e = 0.8$. While negative values of $\Psi(\vartheta, \varphi)$ are physically meaningless, the stresses and other such quantities have reasonable values since they are calculated by 'averaging' over the collision sphere.

5. Concluding remarks

In the present paper we have presented two kinetic theories for the rapid flows of cohesionless granular materials and applied them to examine in detail the problem of simple shear flow. The first analysis was based upon the use of the *ad hoc* proposal of Savage & Jeffrey (1981) for the collisional pair-distribution function $f^{(2)}$ and was intended for particles of arbitrary inelasticity. The second approach determined $f^{(2)}$ as part of the overall analysis, but was intended for nearly elastic particles. On the basis of extensive comparisons with various other theories and experiments it was found that the simple approaches of §3.1 and Jenkins & Savage (1983) which make use of the *ad hoc* $f^{(2)}$ give results which at higher concentrations are surprisingly close to those of the more detailed approach developed in §4 of the present paper. The analysis in §4 was able to distinguish some of the finer details of the overall behaviour and is no doubt more appropriate at low concentrations. Although the analyses based upon the *ad hoc* $f^{(2)}$ of Savage & Jeffrey (1981) give fairly good estimates of the 'collisional' contributions to the constitutive equations they cannot predict most of the 'kinetic' contributions.

There are a number of ways in which the present analysis in §4 could be extended. The particles have been assumed to be smooth, uniform spheres. The effects of surface roughness, particle spin and rotary inertia are presently under investigation. Shen (1981) has already made an attempt to incorporate the effects of the interstitial fluid into a granular-flow theory, and it certainly would be worthwhile to try to treat these effects within the context of the present analyses.

The appropriate boundary conditions for flows adjacent to solid boundaries and 'free' surfaces require further investigation. For example, it is of interest to determine how velocity slip and the granular temperature and its gradient might depend upon the wall roughness size and inelasticity. It should be possible to examine such problems in the spirit of the present analyses.

Further efforts should be devoted to the inclusion of rate-independent stress contributions which can result from enduring contacts between particles. Such stresses were neglected in all the above theories which in effect assume instantaneous binary collisions between particles. These effects are evident in some experimental results at high concentrations (Savage & Sayed 1980) which depart significantly from the (shear-rate)^2 dependence predicted by the collisional momentum-transfer theories mentioned in the present paper.

Grateful acknowledgement is made to the Natural Sciences and Engineering Research Council of Canada (NSERC) for support of this work through an NSERC operating grant. C. K. K. Lun was supported by an NSERC Postgraduate Scholarship and S. B. Savage was supported by an NSERC International Collaborative Research Grant. S. B. Savage is indebted to Professor G. K. Batchelor for his hospitality at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, where the work for this paper was initiated.

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